

Optimal Patent Length and Patent Width for an Economy with Creative Destruction and Non-Diversifiable Risk

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Abstract

This study examines optimal public policy in a product cycle model where R&D firms innovate and imitate and households face non-diversifiable risk. The government controls product cycles by two policy instruments: *patent length*, i.e. the expected time an innovation is imitated, and *patent width*, i.e. the innovator's profit after a successful imitation relative to that before. The main results are the following. An increase in patent length or patent width slows down economic growth. The more patient or the less risk averse the households, the longer and narrower the optimal patents.

Journal of Economic Literature: L11, L16, O31, O34

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1 Introduction

A patent is an innovation protected by the authorities. In a growth model of creative destruction, firms can step forward in the quality ladders of technology by investing in R&D.¹ If imitation is possible as well, then economic growth is subject to product cycles as follows. Through the development of new products, an innovator achieves a temporary advantage earning monopoly profits. This advantage ends when an imitator succeeds in copying the innovation, enters the market and starts competing with the innovator. I define *patent length* as the time the patentee earns the full monopoly profit,² and *patent width* (or breadth) as the innovator's relative profit after and before the entry of a successful imitator.³ The purpose of this paper is to find the welfare-maximizing patent shape (length + width) for an economy with creative destruction and non-diversifiable risk.

In models with no uncertainty, patent length is commonly the expiration time of the patent. In product cycle models, however, this is not appropriate, because any patent is likely replaced by a new innovation before it would expire. In that case, it is better to define patent length as the actual time the patentee can exploit the monopoly profit.

In product cycle models, the assumption of the diversifiable risk simplifies the analysis considerably: Firms can borrow any amount for R&D at a given interest rate and households are protected from uncertainty through diversification in the market portfolio. In that case, the optimal patent length and patent width are obtained by maximizing the present value of a R&D firm independently of the households' behavior. In the literature assuming implicitly that risk is diversifiable, the optimal patent shape is highly sensitive to the firms' cost structure. Some papers claim that "long and narrow" patents are superior to "short and wide" patents, while some others claim vice versa.⁴ The problem with the assumption of diversifiable risk, however,

¹Cf. Grossman and Helpman (1991) (in ch. 4), and Aghion and Howitt (1998).

²This definition is also equivalent to Horii and Iwaisako's (2007) concept of Intellectual Property Rights (IPR).

³In other words, the more the patentee's profit falls at the occurrence of imitation, the narrower the patent. This definition is in line with cf. Denicolo (1996), Takalo (1998) and Kanninen and Stenbacka (2000).

⁴Cf. Denicolo (1996), Takalo (1998) and Kanninen and Stenbacka (2000).

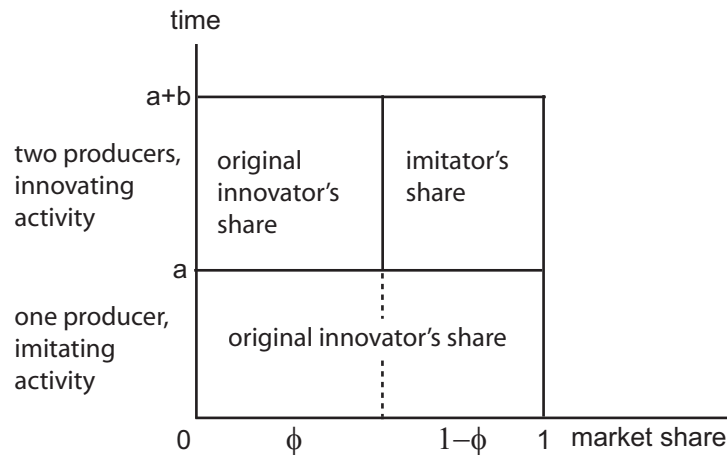


Figure 1: Stages of the product cycle.

is that it contradicts the entire literature of venture capital which supposes that firms cannot borrow without collateral and use their immaterial property (e.g. patents) as collateral.⁵ Where households cannot wholly diversify their investment risk, firms finance their R&D through issuing shares and households purchasing these shares face the uncertainty associated with investment. In that case, patent policy depends on the households' behavior as well. For all the reasons given above, it would be instructive to examine the optimal patent shape in an economy with non-diversifiable risk.

The structure of a product cycle model is characterized in Fig. 1. Let 0 be the starting time at which an innovation occurs, a the time at which the innovation is imitated, $a + b$ the time at which an innovation of the next generation occurs. The innovator possesses the whole market during the imitative period $[0, a)$ and the share $\phi \in (0, 1)$ of the market during the innovative period $[a, a + b)$, while the imitator possesses the share $1 - \phi$ of the market during $[a, a + b)$. If imitation is serially uncorrelated, then the probability of a successful imitation is equal to the inverse of the time of the imitation, $1/a$. If innovation is serially uncorrelated, then the probability of a successful innovation is equal to the inverse of the time of the innovation, $1/b$. In this framework, patent length – i.e. the expected time in which an innovation will be imitated – is given by a . if profit is proportional to

⁵A nice summary of this literature is given by Gompers and Lerner (1999).

the market share, then patent width – i.e. the innovator’s relative profit (= relative market share) after and before imitation – is given by ϕ . I shall maximize welfare by patent length and patent width in the presence of non-diversifiable risk.

In an earlier paper of mine, I examine the growth effects of competition in a product cycle model (Palokangas 2008). In that paper, I extend Wälde’s (1999a, 1999b) growth model with non-diversifiable risk for a multi-sector economy and incorporate into it Segerstrom’s (1991) and Mukoyama’s (2003) ideas on product cycles and cumulative technology, with the following results. A little intensity of competition is growth diminishing. Only if the intensity of competition exceeds a critical level, its increase enhances growth. In contrast to Palokangas (2008), I assume in this document that the producers share the market among themselves and that there is a benevolent government that regulates the productivity of imitation and the innovator’s market share.

If competition had no direct effect on welfare, the government would have no incentives to promote competition in patent policy. In order to construct such a direct effect, I adopt Ethier’s (1982) assumption that households are better off, if the same amount of resources are used to produce a broader variety of products. In that case, competition in any industry raises every household’s welfare through an increased number of firms and products.

In line with Aghion et al. (2001), I specify household preferences so that labor supply is infinitely elastic, for simplicity. Aghion et al. (1997) show that with inelastic labor supply wage movements mitigate the effect of competition on growth. On the one hand, product market competition tends to increase the demand for manufacturing workers within each industry, but on the other hand the demand for R&D workers should simultaneously go up as a result of the increased incentive for the firms to escape competition. The resulting upward pressure on wages reduce the innovators’ incremental rents and their incentives to innovate. The only difference is that Aghion et al. (2001) use a utility function that is logarithmic in consumption, but linear in labor supply, while I use a utility function with a constant rate of risk aversion. In that case, I have to introduce infinitely elastic labor supply through the assumption that the disutility of employment is in fixed proportion to the “standard of living” (= the average consumption).

As a result of all these modifications, I obtain optimal patent length and patent width for an economy with non-diversifiable risk and product cycles. The remainder of this paper is organized as follows. Section 2 presents the structure of the model, section 3 proves the existence of the equilibrium and section 4 constructs the product cycle. Sections 5 establishes welfare-maximizing policy as functions of the rate of time preference.

2 The model

2.1 The markets

There is a large number of households that are placed evenly over the limit $[0, 1]$. Household $\iota \in [0, 1]$ consumes the C_ι units of the final good and supplies N_ι labor units. Because in the model there is no money that would pin down the nominal price level at any time, it is convenient to normalize the households' total spending in consumption at unity:

$$Py = 1, \quad y \doteq \int_0^1 C_\iota d\iota, \quad (1)$$

where y is aggregate consumption and P the consumption price. Aggregate labor supply (= all households' labor supply) is equal to total labor devoted to production, x , and total labor devoted to R&D., l :

$$\int_0^1 N_\iota d\iota = x + l. \quad (2)$$

Competitive firms produce the consumption good from a great number of intermediate goods that are evenly placed over the limit $[0, 1]$. In industry j , there are n_j firms that produce one unit of the same intermediate good $j \in [0, 1]$ from one labor unit. Aggregate consumption y is then produced through Cobb-Douglas technology as follows:

$$\ln y = \int_0^1 \ln(B_j x_j) dj, \quad x_j = \sum_{\kappa=1}^{n_j} x_{j\kappa}, \quad (3)$$

where B_j is the productivity parameter in industry j , x_j the quantity of intermediate good j , n_j the number of firms in industry j and $x_{j\kappa}$ the output

of firm κ in industry j . Total labor devoted to production is defined by

$$x = \int_0^1 x_j dj. \quad (4)$$

It is challenging to specify the producers' strategic behavior in a product cycle model. Mukoyama (2003) assumes Bertrand competition, under which oligopolists earn zero profits in equilibrium. In an earlier paper of mine, however, I show that there will be no growth without profits in a product cycle model with non-diversifiable risk (Palokangas 2008). To obtain positive profits in equilibrium, one could assume that the oligopolists' products are imperfect substitutes (cf. Aghion et al. 1997, 2001, Palokangas 2008). Because this would excessively complicate the model, then, following Segerstrom (1991), I opt for the assumption that oligopolists cooperate in price settlement. This specification has two benefits. First, Segerstrom shows that cooperation is a stable equilibrium in a product cycles model. Second, any number of cooperating oligopolists charge the monopoly price. If the oligopolists charged a lower price than a monopoly, the model would again become excessively complicated. Therefore:

Assumption 1 *The producers of an intermediate good share the market and cooperate in price settlement. The entry of the $(n_j + 1)$ th producer in the market decreases the market share of all the n_j incumbent producers.*

Although the oligopolists cooperate, they do not share the market equally. The first producer, who is the innovator while the other are imitators in the market, has some advantages (reputation, networking, etc.) by which it captures a bigger market share than the others. I assume that government regulations affect this market sharing, so that *patent width*, i.e. the innovator's relative profit after and before a successful imitation, can be defined as the government's policy parameter.

Because R&D firms finance their expenditure by issuing shares and the households save only in these shares, aggregate income is equal to the value of consumption, Py , plus wages paid in R&D, wl , where w is the wage and l labor devoted to R&D. Given (1), it is then true that

$$\int_0^1 A_t dt = wl + Py = wl + 1, \quad (5)$$

where A_ι is the income of household $\iota \in [0, 1]$ and $\int_0^1 A_\iota d\iota$ aggregate income. All households are risk averters and share the same preferences.

2.2 Preferences

The utility function of a single household $\iota \in [0, 1]$ is based on three principles. First, in order to introduce the rate of risk aversion as a parameter of the model, I assume the following:

Assumption 2 *All households share the same preferences in which the rate of time preference, $\rho > 0$, and the rate of relative risk aversion, $\epsilon \in (0, 1)$, are constants.*

Second, I adopt Ethier's (1982) assumption that households are better off, if the same amount of resources are used to produce a broader variety of products. I specify this assumption in the form that an increase in the number n_j of firms in any industry $j \in [0, 1]$ raises every household's welfare:

Assumption 3 *The level of a household's utility is an increasing function of the number of firms in the economy, $\int_0^1 n_j dj$:*

$$f\left(\int_0^1 n_j dj\right), \quad f' > 0, \quad f \text{ strictly concave.} \quad (6)$$

If households did not benefit from using the same amount of resources in a broader variety of products, it would be socially optimal to produce all goods by monopolies only.⁶

Third, I introduce infinitely elastic labor supply as follows:⁷

Assumption 4 *In equilibrium, the disutility of employment in terms of consumption – when utility is held constant – is in fixed proportion ξ to the “standard of living” (= the households' average consumption in the economy, y).*

⁶In subsection 5.1, the effect (i) would be missing and the scale of competition, β , would be equal to zero.

⁷Aghion et al. (2001) introduce infinitely elastic labor supply by the assumption that utility is logarithmic in consumption but linear in labor supply. Because I change logarithmic utility $\epsilon = 1$ into the constant rate $\epsilon \in (0, 1)$ of relative risk aversion, I have to replace the linearity of labor supply in utility by assumption 4.

The following temporary utility function u satisfies assumption 4:

$$\begin{aligned} &u(C_\iota, N_\iota, y), \quad \partial u / \partial C_\iota > 0, \quad \partial u / \partial N_\iota < 0, \\ &u \text{ strictly concave,} \quad u \text{ linearly homogeneous in } (C_\iota, y), \\ &\left. \frac{dC_\iota}{dN_\iota} \right|_{u \text{ constant}, C_\iota=y} = - \left(\frac{\partial u}{\partial N_\iota} / \frac{\partial u}{\partial C_\iota} \right)_{C_\iota=y} = \xi y, \end{aligned} \quad (7)$$

where C_ι is household ι 's consumption, N_ι its labor supply, $\partial C_\iota / \partial N_\iota$ its disutility of employment, y average consumption and $\xi > 0$ a constant.⁸

Given assumptions 2, 3 and 4, I can write household ι 's inter-temporal utility beginning at time T as follows:

$$U(C_\iota, T) = E \int_T^\infty f \left(\int_0^1 n_j dj \right) u(C_\iota, N_\iota, y)^{1-\epsilon} e^{-\rho(t-T)} dt, \quad (8)$$

where t is time and E the expectation operator.

2.3 Research and development (R&D)

The productivity parameter in industry j [cf. (3)] is determined by

$$B_j \doteq \mu^{\tau_j}, \quad \mu > 1, \quad (9)$$

where μ is a parameter and τ_j an index of technology in industry j . The invention of a new technology in industry j raises the index τ_j by one and the level of productivity by $\mu > 1$.

In any industry, there can be either *innovative R&D* that aims at creating a new state-of-the-art product in the industry, or *imitative R&D* that aims at creating a close substitute for the incumbent state-of-the-art product at the same level of technology. I denote the set of imitative industries by Θ and that of innovative industries by $[0, 1] \setminus \Theta$.

I assume that all firms doing innovative R&D learn from each others. In each innovative industry $j \in [0, 1] \setminus \Theta$, firms $\ell \in \{1, \dots, n_j\}$ employ labor $l_{j\ell}$ in innovative R&D. This produces the total *spillover effect*

$$\int_{k \in [0, 1] \setminus \Theta} \sum_{\ell=1}^{n_k} l_{k\ell} dk = \int_{k \notin \Theta} \sum_{\ell=1}^{n_k} l_{k\ell} dk$$

⁸Examples of the functions (7) are $u = (C_\iota^{1-\theta} - \xi y^{1-\theta} N_\iota) / (1 - \theta)$ and $u = (C_\iota - \xi y N_\iota)^{1-\theta} / (1 - \theta)$, where $\theta \in (0, 1) \cup (1, \infty)$ is a constant.

for all firms doing innovative R&D. When firm κ in industry j innovates, its technological change follows a Poisson process $q_{j\kappa}$ in which the arrival rate of innovations, $\Lambda_{j\kappa}$, is given by

$$\Lambda_{j\kappa} = \lambda l_{j\kappa}^{1-\delta} \left(\int_{k \notin \Theta} \sum_{\ell=1}^{n_k} l_{k\ell} dk \right)^\delta, \quad \lambda > 0, \quad 0 < \delta < 1, \quad (10)$$

where λ and δ are constants. During a short time interval dt , there is an innovation $dq_{j\kappa} = 1$ in firm κ with probability $\Lambda_{j\kappa} dt$, and no innovation $dq_{j\kappa} = 0$ with probability $1 - \Lambda_{j\kappa} dt$.

The specification (10) has the following useful property. In the symmetric equilibrium where all innovative firms employ the same amount of labor (i.e. $l_{k\ell} = l_{j\kappa}$ for all $k \notin \Theta$ and ℓ) and each innovative industry has the same number of firms (i.e. $n_k = n_j$ for all $k \notin \Theta$), the arrival rate of innovations per firm, $\Lambda_{j\kappa}$, is in fixed proportion λ to labor input per innovative firm, $l_{j\kappa}$, times the spillover effect, $(n_j \beta)^\delta$, which is an increasing function of the number of innovative firms in the economy, $n_j \beta$.

When firm κ in industry j imitates, its technological change follows a Poisson process $Q_{j\kappa}$ in which the arrival rate of imitations is in fixed proportion λ/a to the firm's own labor input $l_{j\kappa}$:⁹

$$\Gamma_{j\kappa} = (\lambda/a) l_{j\kappa}, \quad a > 0. \quad (11)$$

During a short time interval dt , there is an imitation $dQ_{j\kappa} = 1$ with probability $\Gamma_{j\kappa} dt$, and no imitation $dQ_{j\kappa} = 0$ with probability $1 - \Gamma_{j\kappa} dt$. The relative productivity between imitative and innovative R&D, a , characterizes *patent length*. The more government regulations hamper imitation, the bigger a and the longer time it takes to produce a successful imitation for an invention. Given this, I define patent length a as the government's policy parameter.

Each R&D firm distributes its profit among those who had financed it in proportion to their investment in the firm. Because both innovation and imitation follow a Poisson process, the values of shares in R&D projects are random variables and household $\iota \in [0, 1]$ maximizes its utility (8) subject to the random development of these values.

⁹I ignore spillover effect for imitative R&D, for simplicity. When all imitative firms are subject to constant returns to scale with respect to their own input, they all behave as if there were a single imitative firm in each industry.

3 The steady-state equilibrium

In this section, I prove the existence of the following equilibrium:

Definition. *The economy is in a stationary-state equilibrium, if the following properties are satisfied:*

- (i) *The industries j are run either by monopolies ($n_j = 1$) or duopolies ($n_j = 2$). Non-producing outsiders imitate to enter any of the monopoly industries and the incumbent duopolists innovate to become a monopoly in the same industry. The profits of a typical monopoly and a typical duopolist are constant over time.*
- (ii) *The proportions of monopoly and duopoly industries in the economy (denoted α and β , respectively) are constants over time. Every time a new superior-quality product is discovered in some industry, changing this from a duopoly into a monopoly, imitation must occur in some other industry, changing this from a monopoly into a duopoly.*
- (iii) *The average growth rate of consumption, g , the wage w , total labor in manufacturing, x , and total labor in R&D, l , the labor input η of a typical innovative firm in R&D and the labor input ψ of a typical imitative firm in R&D are constants over time.*

3.1 The manufacturing of goods

The representative consumption-good firm maximizes its profit

$$\Pi^c \doteq Py - \int_0^1 p_j x_j dj$$

subject to technology (3), given the output price P and the input prices p_j . Noting (1), this implies

$$\Pi^c = 0, \quad p_j = P \frac{\partial y}{\partial x_j} = P \frac{y}{x_j} = \frac{1}{x_j} \text{ for all } j. \quad (12)$$

All intermediate-good firms produce one unit of their output from one labor unit. The product of the newest generation provides exactly the constant $\mu > 1$ times as many services as that of earlier generation. A firm of earlier

generation earns the profit $\Pi_j^o = (p_j^o - w)x_j^o$, where p_j^o is its output price and x_j^o its output. Every firm with the newest technology pushes and keeps the firms with older technology out of the market by choosing its price p_j so that these earn no profit, $\Pi_j^o = 0$ and $p_j^o = w$. This yields $p_j/\mu = p_j^o = w$. This, (3), (4) and (12) yields the equilibrium conditions:

$$p_j = \mu w, \quad x_j = \frac{1}{p_j} = \frac{1}{\mu w} = x, \quad \Pi_j = (p_j - w)x_j = \Pi \doteq 1 - \frac{1}{\mu} > 0. \quad (13)$$

Because a successful innovation crowds out all incumbent producers in a market, the innovator is always the first producer, while the later entrants are imitators. Given assumption 1, it is then true that:

- (a) The innovator will earn the constant profit Π as long as it remains the monopoly producer in the industry. Because a household holds the share of all firms in its same portfolio, it does not invest in innovative R&D in the monopoly industries.
- (b) If anyone invests in imitative R&D to enter a monopoly industry j , then its prospective profit is Π_{j2} , but if it does that (with the same cost) to enter an industry j with $\kappa > 1$ producers, then its prospective profit is smaller than Π_{j2} . Thus, it invest in imitative R&D only to enter a monopoly industry, but not to enter an oligopoly industry. This means that there can be at most two producers in an industry.

From (a) and (b) above it follows that in equilibrium there are only monopoly industries with imitative R&D or duopoly industries with innovative R&D. This and (13) *prove the property (i) of a stationary-state equilibrium.*

I denote the set of monopoly industries by $\Theta \subset [0, 1]$. The proportions of duopoly and monopoly industries (β and α , respectively) are then given by

$$\beta \doteq \int_{j \notin \Theta} dj, \quad \alpha = \int_{j \in \Theta} dj = 1 - \beta. \quad (14)$$

Thus, *the property (ii) of a stationary-state equilibrium is proven.* Tang and Wälde (2001) call the proportion of duopoly industries, β , as the *scale of competition*. The profits in the economy are determined as follows:

Proposition 1 *In monopoly industries $j \in \Theta$, the innovator earns the entire monopoly profit Π , while in duopoly industries $j \notin \Theta$, the innovator earns a*

smaller profit $\phi\Pi$ and the imitator earns the rest of the monopoly profit in the industry, $(1 - \phi)\Pi$, where $\phi \in [0, 1]$ is the patent width parameter determined by the government.

3.2 Economic growth

According to the properties (i) and (ii) of a stationary-state equilibrium, duopolists labeled 1 and 2 innovate and none imitates in duopoly industries $j \notin \Theta$, while outsiders imitate and none innovates in monopoly industries $j \in \Theta$. Because according to technology (11) imitation yields constant returns to scale, all outsiders in monopoly industry $j \in \Theta$ behave as if there were a single outsider firm labeled 0. The structure of industries is given by Fig. 2.

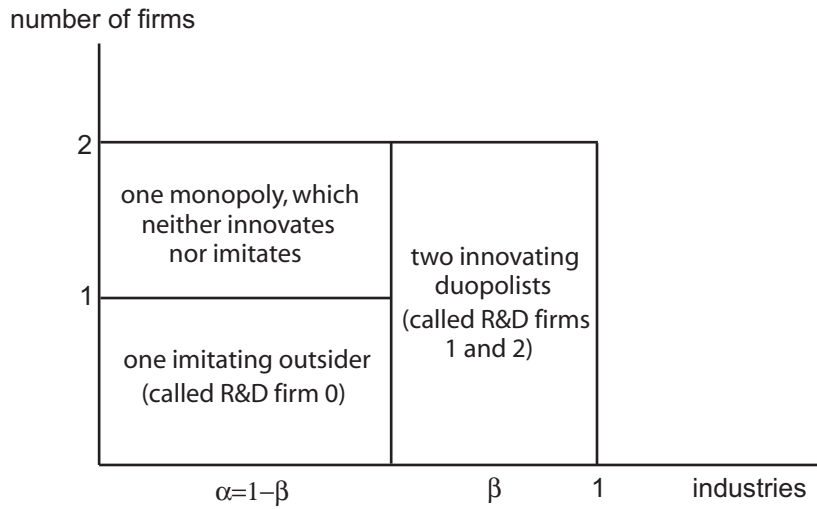


Figure 2: Competition and the number of firms in the economy.

In duopoly industries $j \notin \Theta$ the two producers employ $l_{j1} + l_{j2}$ and in monopoly industries $j \in \Theta$ the outsider employs l_{j0} labor units in R&D. Total employment in R&D, l , is the sum of all firms' employment in R&D:

$$l \doteq \int_{j \notin \Theta} (l_{j1} + l_{j2}) dj + \int_{j \in \Theta} l_j dj. \quad (15)$$

Given (9), the average productivity in the economy, B , is defined as a function

of the technologies τ_j of all industries $j \in [0, 1]$ as follows:

$$\ln B \doteq \int_0^1 \ln B_j dj = (\ln \mu) \int_0^1 \tau_j dj. \quad (16)$$

The arrival rate of innovations in duopoly industry $j \notin \Theta$ is the sum of the arrival rates of both duopolists in that industry, $\Lambda_{j1} + \Lambda_{j2}$ [Cf., (10)]. Because only duopoly industries $j \notin \Theta$ innovate, then the average growth rate of the average productivity $B(\{t_k\})$ in the stationary state is given by

$$g \doteq (\ln \mu) \int_0^1 \Pr(\tau_j \text{ increases by one}) dj = (\ln \mu) \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj, \quad (17)$$

where $\Pr(\cdot)$ denotes the probability.

3.3 Innovation and imitation

In monopoly industry $j \in \Theta$ outsider 0 and in industry $j \notin \Theta$ duopolists 1 and 2 issue shares to finance their labor expenditure in R&D. Because the households $\iota \in [0, 1]$ invest in these shares, one obtains

$$\int_0^1 S_{\iota j 0} d\iota = wl_{j0} \text{ for } j \in \Theta, \quad \int_0^1 S_{\iota j \kappa} d\iota = wl_{j\kappa} \text{ for } \kappa \in \{1, 2\} \text{ and } j \notin \Theta, \quad (18)$$

where wl_{j0} is the imitative expenditure of outsider 0 in monopoly industry $j \in \Theta$, $wl_{j\kappa}$ the innovative expenditure of duopolist $\kappa \in \{1, 2\}$ in industry $j \notin \Theta$, $S_{\iota j 0}$ household ι 's investment in outsider firm 0 in monopoly industry $j \in \Theta$, $S_{\iota j \kappa}$ household ι 's investment in duopolist κ in industry $j \notin \Theta$, $\int_0^1 S_{\iota j 0} d\iota$ aggregate investment in outsider firm 0 in monopoly industry $j \in \Theta$, and $\int_0^1 S_{\iota j \kappa} d\iota$ aggregate investment in duopolist κ in industry $j \notin \Theta$. Household ι 's relative investment shares in outsiders 0 and duopolists $\kappa \in \{1, 2\}$ are

$$i_{\iota j 0} \doteq \frac{S_{\iota j 0}}{wl_{j0}} \text{ for } j \in \Theta; \quad i_{\iota j \kappa} \doteq \frac{S_{\iota j \kappa}}{wl_{j\kappa}} \text{ for } j \notin \Theta. \quad (19)$$

When household ι has financed a successful R&D firm, it acquires the right to the firm's profit in proportion to its relative investment share. Noting proposition 1, the profit sharing can then be characterized as follows:

- $s_{\iota j \kappa}$ household ι 's profit from duopolist $\kappa \in \{1, 2\}$ in industry $j \notin \Theta$;
- $i_{\iota j \kappa}$ household ι 's investment share in duopolist $\kappa \in \{1, 2\}$ in industry $j \notin \Theta$ [cf. (19)];
- Π the profit that duopolist $\kappa \in \{1, 2\}$ in industry $j \notin \Theta$ shall earn after innovation has changed it into a monopoly;
- $\Pi i_{\iota j \kappa}$ the profit that household ι shall get from duopolist $\kappa \in \{1, 2\}$ in industry $j \notin \Theta$ after innovation has changed this into a monopoly;
- $s_{\iota j 0}$ household ι 's profit from outsider 0 in industry $j \in \Theta$;
- $i_{\iota j 0}$ household ι 's investment share in outsider 0 in industry $j \in \Theta$ [cf. (19)];
- $\phi \Pi$ the profit that the innovator in industry $j \in \Theta$ shall earn after a successful imitation of its product;
- $(1 - \phi) \Pi$ the profit that outsider 0 in industry $j \in \Theta$ shall earn after imitation has changed it as the second duopolist;
- $(1 - \phi) \Pi i_{\iota j 0}$ the profit that household ι shall get from outsider 0 in industry $j \in \Theta$ after imitation has changed it into the second duopolist.

The changes in the profits of firms in industry j are functions of the increments $(dq_{j1}, dq_{j2}, dQ_{j0})$ of Poisson processes (q_{j1}, q_{j2}, Q_{j0}) as follows:¹⁰

$$\begin{aligned}
ds_{\iota j \kappa} &= (\Pi i_{\iota j \kappa} - s_{\iota j \kappa}) dq_{j \kappa} - s_{\iota j \kappa} dq_{j(\zeta \neq \kappa)} \quad \text{when } j \notin \Theta; \\
ds_{\iota j 0} &= [(1 - \phi) \Pi i_{\iota j 0} - s_{\iota j 0}] dQ_{j0} \quad \text{when } j \in \Theta; \\
ds_{\iota j 1} &= (\phi \Pi i_{\iota j 1} - s_{\iota j 1}) dQ_{j0} \quad \text{when } j \in \Theta.
\end{aligned} \tag{20}$$

These functions can be explained as follows. If a household invests in innovative duopolist κ in industry $j \notin \Theta$, then, in the advent of a success for that duopolist, $dq_{j \kappa} = 1$, the amount of its share holdings rises up to $\Pi i_{\iota j \kappa}$, i.e. $ds_{\iota j \kappa} = \Pi i_{\iota j \kappa} - s_{\iota j \kappa}$, but in the advent of success for the other duopolist $\zeta \neq \kappa$, its share holdings in duopolist κ fall down to zero, i.e. $ds_{\iota j \kappa} = -s_{\iota j \kappa}$. If a household invests in imitative outsider 0 in monopoly

¹⁰This extends the idea of Wälde (1999a, 1999b).

industry $j \in \Theta$, then, in the advent of a success for outsider 0, $dQ_{j0} = 1$, the amount of its share holdings in that outsider 0 rises up to $(1 - \phi)\Pi i_{lj0}$, i.e. $ds_{lj0} = (1 - \phi)\Pi i_{lj0} - s_{lj0}$, but the amount of its share holdings in incumbent monopoly 1 falls down to $\phi\Pi i_{lj1}$, i.e. $ds_{lj1} = \phi\Pi i_{lj1} - s_{lj1}$.¹¹

3.4 Households

Because investment in shares in R&D firms is the only form of saving in the model, the budget constraint of household ι is given by

$$A_\iota = PC_\iota + \int_{j \in \Theta} S_{lj0} dj + \int_{j \notin \Theta} (S_{lj1} + S_{lj2}) dj, \quad (21)$$

where A_ι is the household's total income, C_ι its consumption, P the consumption price, S_{lj0} the household's investment in outsider firm 0 in monopoly industry $j \in \Theta$, and $S_{lj\kappa}$ the household's investment in duopolist κ in industry $j \notin \Theta$. Household ι 's total income A_ι consists of its wage income (= the wage w times its labor supply N_ι) wN_ι , its profits s_{lj1} from the monopoly in each industry $j \in \Theta$ and its profits s_{lj1} and s_{lj2} from the duopolists 1 and 2 in each industry $j \notin \Theta$. This yields

$$A_\iota = wN_\iota + \int_{j \in \Theta} s_{lj1} dj + \int_{j \notin \Theta} (s_{lj1} + s_{lj2}) dj. \quad (22)$$

Household ι maximizes its utility (8) by its investment, $\{S_{lj0}\}$ for $j \in \Theta$ and $\{S_{lj1}, S_{lj2}\}$ for $j \notin \Theta$, subject to its budget constraint (21), the stochastic changes (20) in its profits, the composition of its income, (22), and the determination of its relative investment shares, (19), given the arrival rates $\{\Lambda_{j\kappa}, \Gamma_{j0}\}$, the wage w and the consumption price P . In the Appendix, this maximization problem is solved by dynamic programming, with the following results.¹² In the households' stationary equilibrium in which the allocation of resources is invariable across technologies, it is true that

$$w \text{ and } x \text{ are constants,} \quad (23)$$

$$\beta = \frac{1}{2} \left(\frac{1 - \phi}{\mu^{1-\epsilon} a} \right)^{1/\delta} \quad \begin{array}{ll} l_{j\kappa} = \eta, & \text{for } j \notin \Theta, \\ l_{j0} = \psi \doteq (l - 2\beta\eta)/(1 - \beta) & \text{for } j \in \Theta, \end{array} \quad (24)$$

¹¹Household ι knows that the two producers profits must sum up to Π . Because it does not invest in incumbent monopoly 1, the investment share i_{lj1} does not change.

¹²A detailed proof will be delivered to a reader on request. The dynamic program is similar to that in Palokangas (2008).

$$\rho + \underbrace{\frac{1 - \mu^{1-\epsilon}}{\ln \mu}}_g = \frac{(2\beta)^\delta \lambda \Pi \mu^{1-\epsilon}}{(1 + wl)w}, \quad (25)$$

$$\Lambda_{j\kappa} = (2\beta)^\delta \lambda \eta \text{ for } j \notin \Theta \text{ and } \kappa \in \{1, 2\}, \quad g = (2^{1+\delta} \lambda \ln \mu) \beta^{1+\delta} \eta, \quad (26)$$

where η (ψ) is the labor input of a single innovative (imitative) firm. Given (24) and (26), *the property (iii) of a stationary-state equilibrium is proven.*

In (23), the wage w is fixed, because the disutility of employment is proportional to average consumption y . This and the firm's equilibrium conditions (13) lead to fixed output x per industry. Results (24) can be explained as follows. An increase in β above the equilibrium increases the spillover of knowledge and the productivity of R&D for each innovative firm. Consequently, there will be more innovations that change duopoly industries into monopolies and β will fall. A decrease in β below the equilibrium decreases the spillover of knowledge and the productivity of R&D for each innovative firm. Consequently, there will be less innovations that change duopoly industries into monopolies and β will rise. Thus, there exists a stable equilibrium for the proportion of innovative duopoly industries, β . With longer or wider patents (i.e. a bigger a or ϕ), there are more incentives to invest in innovative firms to escape competition. With higher investment per innovative firm, more duopolies change into monopolies and the proportion of duopoly industries, β , falls. Thus, the equilibrium level of β falls with both a and ϕ .

According to (25), a household's subjective discount factor

$$\rho + \frac{1 - \mu^{1-\epsilon}}{\ln \mu} g \quad (27)$$

is equal to the marginal rate of return to savings,

$$\frac{1}{2} \left(\frac{1 - \phi}{\mu^{1-\epsilon} a} \right)^{1/\delta} \quad (28)$$

A higher proportion β of innovative duopoly industries increases the spillover of knowledge, the productivity of R&D for each innovative firm and the marginal rate of return to savings, (28). With more labor in R&D (i.e. a higher l), the marginal product of R&D falls. Because households invest their savings in R&D, then the marginal rate of return to savings, (28), falls as well. In (26), the growth rate g is in fixed proportion to the proportion of innovative industries, β , directly and through spillovers β^δ .

4 The product cycle

Differentiating the logarithm of the first-equation (25) totally, and noting $\mu > 1$, one obtains employment in R&D, l , as the following function:

$$\begin{aligned}
 l(g, \beta, \rho, \epsilon), \quad \frac{\partial l}{\partial \beta} &= \frac{\delta}{\beta} \left(\frac{1}{w} + l \right) > 0, \quad \frac{\partial l}{\partial g} = \underbrace{\frac{1 - \mu^{1-\epsilon}}{\ln \mu}}_{-} \underbrace{\frac{\partial l}{\partial \rho}}_{-} > 0, \\
 \frac{\partial l}{\partial \rho} &= - \left(\rho + \frac{1 - \mu^{1-\epsilon}}{\ln \mu} g \right)^{-1} \left(\frac{1}{w} + l \right) < 0, \\
 \frac{\partial l}{\partial \epsilon} &= - \left[\underbrace{\ln \mu}_{+} + \underbrace{\left(\rho + \frac{1 - \mu^{1-\epsilon}}{\ln \mu} g \right)^{-1}}_{+} \underbrace{g \mu^{(1-\epsilon)}}_{+} \right] \left(\frac{1}{w} + l \right) < 0. \tag{29}
 \end{aligned}$$

Given the property (ii) of the stationary-state equilibrium, the rate at which industries leave the group of duopoly industries $k \notin \Theta$ in a small interval dt , $\beta(\Lambda_{j1} + \Lambda_{j2})dt$, is then equal to the rate at which the industries leave the group of monopoly industries $j \in \Theta$, $\alpha\Gamma_{j0}dt$ in that interval dt :

$$\beta(\Lambda_{k1} + \Lambda_{k2}) = \alpha\Gamma_{j0} \text{ for } k \notin \Theta \text{ and } j \in \Theta. \tag{30}$$

Given equations (11), (24), (26) and (30), one obtains

$$1 = \frac{\Lambda_{k1} + \Lambda_{k2}}{\alpha\Gamma_{j0}/\beta} = \frac{(2\beta)^\delta a\eta}{(1-\beta)l_{j0}/\beta} = \frac{(2\beta)^\delta a\eta}{(1-\beta)\psi/\beta} = \frac{(2\beta)^\delta a\eta}{l/\beta - 2\eta}.$$

From this, (26) and (29) it follows that labor per innovative firm and the growth rate are determined by $\eta = l(g, \beta, \rho, \epsilon) / \{\beta[2 + (2\beta)^\delta a]\}$ and

$$\begin{aligned}
 g &= (2\lambda \ln \mu) \frac{l(g, \beta, \rho, \epsilon)}{2^{1-\delta}\beta^{-\delta} + a} \doteq J(g, \beta, a, \rho, \epsilon), \\
 \frac{\partial J}{\partial g} &= \frac{g}{l} \frac{\partial l}{\partial g} > 0, \quad \frac{\partial J}{\partial a} < 0, \quad \frac{\partial J}{\partial \beta} = \frac{g}{l} \frac{\partial l}{\partial \beta} + \frac{\delta g \beta^{-\delta-1}}{2^{1-\delta}\beta^{-\delta} + a} > 0. \tag{31}
 \end{aligned}$$

The right-hand equation (31) defines the growth rate g as a function of (a, ρ, ϵ) . Unfortunately, the variable g appears in both sides of the equation, which makes this dependence mathematically ambiguous. This ambiguity can be eliminated by the stability properties of the model. Assume that vector (a, ρ, ϵ) changes so that $J(g, \beta, a, \rho, \epsilon)$ increases.¹³ This raises the

¹³Formally, this can be proven as follows. Assume that an increase in the growth rate g is in fixed proportion $\varpi > 0$ to the perturbation $J(g, \beta, a, \rho, \epsilon) - g$ from the equilibrium: $\dot{g} = \varpi[J(g, \beta, a, \rho, \epsilon) - g]$. This system has a stable equilibrium only if $\partial \dot{g} / \partial g = \varpi[\partial J / \partial g - 1] < 0$, which is equivalent to $\partial J / \partial g < 1$.

growth rate g by the same amount, which generates a further increase $\partial J/\partial g$ in J . If $\partial J/\partial g < 1$, there will be a sequence of dampening increases in g until a new equilibrium is attained. If $\partial J/\partial g > 1$, then there will be ever accelerating increases in g and the system will never end up with an equilibrium. Because the comparative static properties of a constant-growth equilibrium cannot be analyzed by an unstable model,¹⁴ I assume $\partial J/\partial g < 1$. Given $0 < \partial J/\partial g < 1$, (24) and (25), the comparative statics of the equation (31) implies the function

$$g = G(\beta, a, \rho, \epsilon), \quad \frac{\partial G}{\partial a} \doteq \frac{\partial J}{\partial a} \Big/ \left(1 - \frac{\partial J}{\partial g}\right) < 0, \quad \frac{\partial G}{\partial \beta} \doteq \frac{\partial J}{\partial \beta} \Big/ \left(1 - \frac{\partial J}{\partial g}\right) > 0,$$

$$\frac{dG}{da} = \underbrace{\frac{\partial G}{\partial a}}_{-} + \underbrace{\frac{\partial G}{\partial \beta}}_{+} \underbrace{\frac{\partial \beta}{\partial a}}_{-} < 0, \quad \frac{dG}{d\phi} = \underbrace{\frac{\partial G}{\partial \beta}}_{+} \underbrace{\frac{\partial \beta}{\partial \phi}}_{-} < 0. \quad (32)$$

This result can be rephrased as follows:

Proposition 2 *An increase in patent length a or patent width ϕ slows down economic growth (i.e. decreases g).*

According to (32), patent shape affects the growth rate through two channels:

The scale-of-competition effect. With longer or wider patents, there are more incentives to invest in innovative R&D firms to escape competition. With higher investment per innovative firm, more duopolies will end up as monopolies and the proportion of innovating industries (= the scale of competition), β , will fall. This will slow down economic growth.

The direct effect. Assume that patent length a is increased, but patent width ϕ is decreased to hold the proportion of innovating industries, β , constant [cf. (24)]. In that case, there are less imitative firms flowing to the group of innovative firms. Consequently, in equilibrium, there must be less successful innovations transferring firms to the group of imitative firms. With less innovations, the growth rate will be lower.

¹⁴Only in a stable system, a small change of the vector (a, ρ, ϵ) generates a small change in the equilibrium value of the endogenous variable g .

Patent length a promotes economic growth directly and through the scale of competition, but patent width ϕ only through the latter. This difference enables the control of the growth rate g and the scale of competition, β , by patent length a and patent width ϕ .

5 The government

Noting (1), (2), (3), (4), (16), (29) and the symmetry across the households $\iota \in [0, 1]$, one obtains consumption y as:

$$C_\iota = y = xB \text{ for } \iota \in [0, 1], \quad N_\iota = x + l \text{ for } \iota \in [0, 1]. \quad (33)$$

Noting (6), (7), (14), (29) and (33), I define the function

$$\begin{aligned} c(g, \beta, \rho, \epsilon) &\doteq \frac{f(\int_0^1 n_j dj)}{B^{1-\epsilon}} u(C_\iota, N_\iota, y)^{1-\epsilon} = f(1 + \beta)x^{1-\epsilon}u(1, x + l, 1)^{1-\epsilon}, \\ \frac{1}{c} \frac{\partial c}{\partial g} &= \frac{1 - \epsilon}{u} \frac{\partial u}{\partial N_\iota} \frac{\partial l}{\partial g} = \underbrace{\frac{1 - \epsilon}{u}}_+ \underbrace{\frac{\partial u}{\partial N_\iota}}_- \underbrace{\left(\rho + \frac{1 - \mu^{1-\epsilon}}{\ln \mu} g\right)^{-1}}_+ \underbrace{\frac{\mu^{1-\epsilon} - 1}{\ln \mu}}_+ \underbrace{\left(\frac{1}{w} + l\right)}_+ < 0, \\ \frac{1}{c} \frac{\partial c}{\partial \beta} &= \frac{1 - \epsilon}{u} \frac{\partial u}{\partial N_\iota} \frac{\partial l}{\partial \beta} + \frac{f'}{f} = \underbrace{\frac{1 - \epsilon}{u}}_+ \underbrace{\frac{\partial u}{\partial N_\iota}}_- \underbrace{\frac{\delta}{\beta} \left(\frac{1}{w} + l\right)}_+ + \underbrace{\frac{f'(1 + \beta)}{f(1 + \beta)}}_+. \end{aligned} \quad (34)$$

Noting this, a single household's utility function (8) takes the form

$$U(C_\iota, T) = E \int_T^\infty c(g, \beta, \rho, \epsilon) B(\{t_k\})^{1-\epsilon} e^{-\rho(\nu-T)} d\nu. \quad (35)$$

The government maximizes a household's welfare (35) subject to stochastic technological change (10). Noting (24) and (32), the growth rate g and the scale of competition, β , can be controlled by patent length a and patent width ϕ . Thus, the maximization can be carried out by using the growth rate g and the scale of competition, β as control variables. Later on, the optimum can be transformed into the terms of patent and competition policy (a, ϕ).

I denote by $\Upsilon(\{t_k\})$ the value of any industry using current technology t_k , and by $\Upsilon(t_j + 1, \{t_{k \neq j}\})$ the value of industry j using technology $t_j + 1$, when other industries $k \neq j$ use current technology t_k . In each duopoly industry $j \notin \Theta$, the arrival rate of innovations that change technology from t_j to t_{j+1}

is equal to $\Lambda_{j1} + \Lambda_{j2}$, while there are no innovations in monopoly industries $j \in \Theta$. Noting this, the Bellman equation for the government's program is¹⁵

$$\rho \Upsilon(\{t_k\}) = \max_{g, \beta} \mathcal{F}(g, \beta, \rho, \epsilon), \quad \text{where}$$

$$\mathcal{F}(g, \beta, \rho, \epsilon) \doteq \frac{c(g, \beta, \rho, \epsilon)}{B^{\epsilon-1}} + \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) [\Upsilon(t_j + 1, \{t_{k \neq j}\}) - \Upsilon(\{t_k\})] dj.$$

Because in equilibrium technological change is symmetric throughout all innovative industries,

$$\Upsilon(t_j + 1, \{t_{k \neq j}\}) - \Upsilon(\{t_k\}) = \Upsilon(t_l + 1, \{t_{k \neq l}\}) - \Upsilon(\{t_k\}) \text{ for } j \notin \Theta,$$

then, noting (17), this Bellman equation changes into

$$\rho \Upsilon(\{t_k\}) = \max_{g, \beta} \mathcal{F}(g, \beta, \rho, \epsilon), \quad \text{where}$$

$$\mathcal{F}(g, \beta, \rho, \epsilon) = \frac{c(g, \beta, \rho, \epsilon)}{B^{\epsilon-1}} + [\Upsilon(t_l + 1, \{t_{k \neq l}\}) - \Upsilon(\{t_k\})] \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj$$

$$= \frac{c(g, \beta, \rho, \epsilon)}{B(\{t_k\})^{\epsilon-1}} + [\Upsilon(t_l + 1, \{t_{k \neq l}\}) - \Upsilon(\{t_k\})] \frac{g}{\ln \mu}. \quad (36)$$

Provided that the function $c(g, \beta, \rho, \epsilon)$ is strictly concave in (c, g) , the function $\mathcal{F}(g, \beta, \rho, \epsilon)$ is strictly concave in (c, g) as well. In that case, the government's optimum is unique and one can apply comparative statics on public policy.

5.1 The optimal scale of competition

Noting (34) and (36), one obtains

$$\beta = \arg \max_{\beta} \mathcal{F}(g, \beta, \rho, \epsilon) = \arg \max_{\beta} c(g, \beta, \rho, \epsilon). \quad (37)$$

This can be rephrased as:

Proposition 3 *The welfare-maximizing scale of competition, β , maximizes temporary utility c at every moment of time.*

The scale of competition, β , has two opposite effects on temporary utility c :

- (i) It increases the variety of products and temporary utility c .

¹⁵Cf. Dixit and Pindyck (1994).

- (ii) Because duopolies employ more workers in manufacturing than monopolies, a larger proportion of duopolies, β , increases employment in manufacturing and the wage. With a higher wage, output per firm is lower in the production of the consumption good. This decreases consumption and temporary utility c .

The scale of competition, β , maximizes consumption when the opposite effects (i) and (ii) are in balance. The first-order condition corresponding to (37) is $\partial c/\partial\beta = 0$. Given (34), this is equivalent to

$$\frac{\beta f'(1+\beta)}{f(1+\beta)} = - \underbrace{\delta}_{+} \underbrace{\frac{1-\epsilon}{u}}_{-} \underbrace{\frac{\partial u}{\partial N_t} \left(\frac{1}{w} + l\right)}_{+}. \quad (38)$$

5.2 The optimal growth rate

I try the solution that the value function is of the form

$$\Upsilon(\{t_k\}) = cB(\{t_k\})^{1-\epsilon}/\vartheta \quad (39)$$

where ϑ is independent of the endogenous variables of the system. From (9), (16) and (39) it then follows that

$$\frac{\Upsilon(t_j + 1, \{t_{k \neq j}\})}{\Upsilon(\{t_k\})} = \left(\frac{B(t_j + 1, \{t_{k \neq j}\})}{B(\{t_k\})} \right)^{1-\epsilon} = \left(\frac{B_j(t_j + 1)}{B_j(t_j)} \right)^{1-\epsilon} = \mu^{1-\epsilon}. \quad (40)$$

Inserting (39) and (40) into the Bellman equation (36), I obtain

$$\rho = \vartheta + (\mu^{1-\epsilon} - 1) \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj = \vartheta + (\mu^{1-\epsilon} - 1) \frac{g}{\ln \mu}$$

and

$$\vartheta = \rho + \frac{1 - \mu^{1-\epsilon}}{\ln \mu} g. \quad (41)$$

Noting (34), (36), (39), (40) and (41), one obtains

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial g} &= B^{1-\epsilon} \frac{\partial c}{\partial g} + \frac{1}{\ln \mu} [\Upsilon(t_t + 1, \{t_{k \neq t}\}) - \Upsilon(\{t_k\})] \\ &= B^{1-\epsilon} \frac{\partial c}{\partial g} + \frac{\mu^{1-\epsilon} - 1}{\ln \mu} \Upsilon(\{t_k\}) = \frac{\vartheta}{c} \Upsilon(\{t_k\}) \frac{\partial c}{\partial g} + \frac{\mu^{1-\epsilon} - 1}{\ln \mu} \Upsilon(\{t_k\}) \\ &= \left[\frac{\vartheta}{c} \frac{\partial c}{\partial g} + \frac{\mu^{1-\epsilon} - 1}{\ln \mu} \right] \Upsilon(\{t_k\}) = \left[\left(\rho + \frac{1 - \mu^{1-\epsilon}}{\ln \mu} g \right) \frac{1}{c} \frac{\partial c}{\partial g} + \frac{\mu^{1-\epsilon} - 1}{\ln \mu} \right] \Upsilon(\{t_k\}) \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{1-\epsilon}{u} \frac{\partial u}{\partial N_l} \frac{\mu^{1-\epsilon} - 1}{\ln \mu} \left(\frac{1}{w} + l \right) + \frac{\mu^{1-\epsilon} - 1}{\ln \mu} \right] \Upsilon(\{t_k\}) \\
&= \left[\frac{1-\epsilon}{u} \frac{\partial u}{\partial N_l} \left(\frac{1}{w} + l \right) + 1 \right] \frac{\mu^{1-\epsilon} - 1}{\ln \mu} \Upsilon(\{t_k\}) = 0.
\end{aligned}$$

Noting this and (38), one obtains that labor devoted to R&D, l , and the scale of competition, β , are constants:

$$\left[\frac{\epsilon - 1}{u(1, N_l, 1)} \frac{\partial u(1, N_l, 1)}{\partial N_l} \right]_{N_l=x+l} \left(\frac{1}{w} + l \right) = 1, \quad \frac{d \log f}{d\beta} = \frac{\beta f'(1 + \beta)}{f(1 + \beta)} = \delta. \quad (42)$$

These equations fix l and β and yield the following result:

Proposition 4 *The scale of competition, β , should be extended up to the point at which the elasticity of output with respect to β – holding labor input x and the level of productivity, B , constant – is equal to the spillover parameter δ in the production function of the innovative R&D firm. At the optimum, both labor devoted to R&D, \tilde{l} , and the scale of competition, $\tilde{\beta}$, are independent of the rate of time preference, ρ , and the rate of risk aversion, ϵ .*

Noting (42) and (29) yield $l(g, \beta, \rho, \epsilon) = \tilde{l}$. Differentiating this equation totally and noting (29), one obtains

$$g(\rho, \epsilon), \quad \underbrace{\frac{\partial g}{\partial \rho}}_{-} = - \underbrace{\frac{\partial l}{\partial \rho}}_{-} / \underbrace{\frac{\partial l}{\partial g}}_{+} > 0, \quad \underbrace{\frac{\partial g}{\partial \epsilon}}_{-} = - \underbrace{\frac{\partial l}{\partial \epsilon}}_{-} / \underbrace{\frac{\partial l}{\partial g}}_{+} > 0. \quad (43)$$

This result can be rephrased as follows:

Proposition 5 *If the households become more patient (i.e. ρ falls) or less risk averse (i.e. ϵ falls), then the welfare-maximizing growth rate decreases.*

In this result, the most dominating effect is the following. If households become more patient or less risk averse, then the discount factor (27) falls for a given growth rate g . In that case, the growth rate g must fall to keep the discount factor (27) equal to the marginal rate of return to savings, (28) [cf. $\mu > 1$ and (25)].

5.3 The optimal shape of patents

From (31), (42), (43) and proposition 4 it follows that

$$g(\rho, \epsilon) = (2\lambda \ln \mu)\tilde{l}/[2^{1-\delta}\tilde{\beta}^{-\delta} + a],$$

where \tilde{l} and $\tilde{\beta}$ are constants. Solving for a and noting (43) yield the function

$$a(\rho, \epsilon) \doteq \frac{(2\lambda \ln \mu)\tilde{l}}{g(\rho, \epsilon)} - 2^{1-\delta}\tilde{\beta}^{-\delta}, \quad \frac{\partial a}{\partial \rho} < 0, \quad \frac{\partial a}{\partial \epsilon} < 0. \quad (44)$$

Finally, given (24), (44) and $\mu > 1$, one obtains

$$\begin{aligned} \phi(\rho, \epsilon) &\doteq 1 - (2\tilde{\beta})^\delta a(\rho, \epsilon)\mu^{1-\epsilon}, \quad \frac{\partial \phi}{\partial \rho} = -(2\tilde{\beta})^\delta \mu^{1-\epsilon} \frac{\partial a}{\partial \rho} > 0, \\ \frac{\partial \phi}{\partial \epsilon} &= (2\tilde{\beta})^\delta \mu^{1-\epsilon} \left[\underbrace{a \ln \mu}_+ - \underbrace{\frac{\partial a}{\partial \epsilon}}_- \right] > 0. \end{aligned} \quad (45)$$

The results (44) and (45) can be rephrased as follows:

Proposition 6 *The more patient (i.e. the smaller ρ) or the less risk averse (i.e. the smaller ϵ) the households, the longer and narrower the optimal patents (i.e. the bigger a and the smaller ϕ).*

With more patient or less averse households the welfare-maximizing growth rate is lower (cf. proposition 5). This low growth rate can be implemented either long or wide patents (cf. proposition 2). Because patent length a promotes economic growth both directly and through the scale of competition, β , but patent width ϕ only through the latter, it is more efficient to slow down growth by increasing patent length and to hold the scale of competition, β , at the optimal level (cf. proposition 4) by decreasing patent width.

6 Conclusions

This study examines a multi-industry economy in which growth is generated by creative destruction. In each industry, a firm that creates the newest technology by a successful innovation crowds out the other firms with older technologies from the market and becomes the first producer of the industry.

A firm creating a copy of the newest technology starts producing the innovator's product and establishes an innovation race with the first producer. Because systematic investment risk cannot be eliminated by diversification, the households hold the shares of all firms in their portfolios.

Innovations are protected by patents. Some patent regulations increase the expected time a patent will be imitated (i.e. patent length), while the others increase the innovator's market share after a successful imitation (i.e. patent width). With these two instruments, the government can control innovative and imitative R&D, economic growth and social welfare. The main findings are as follows.

An increase in patent length or patent width slows down economic growth. Patent shape affects the growth rate through two channels:

The scale-of-competition effect. With longer or wider patents, there are more incentives to invest in innovative R&D firms to escape competition. With higher investment per innovative firm, more duopolies will end up as monopolies and the proportion of innovating industries (= the scale of competition) will fall. This will slow down economic growth.

The direct effect. Assume that patent length is increased, but patent width is decreased to hold the proportion of innovating industries constant. In that case, there are less imitative firms flowing to the group of innovative firms. Consequently, in equilibrium, there must be less successful innovations transferring firms to the group of imitative firms. With less innovations, the growth rate will be lower.

Patent length promotes economic growth directly and through the scale of competition, but patent width only through the latter. This difference enables that the government can control the growth rate and the scale of competition independently. If the households become more patient or less risk averse, then the welfare-maximizing growth rate decreases. In that case, the discount factor falls for a given growth rate, and the growth rate must fall to keep the discount factor equal to the marginal rate of return to savings.

The more patient or the less risk averse the households, the longer and narrower the optimal patents. In that case, the welfare-maximizing growth rate is low and this low growth rate can be implemented by either long or wide

patents. Because patent length promotes economic growth both directly and through the scale of competition, but patent width only through the latter, it is better to lengthen patents in order to slow down economic growth and to narrow patents in order to hold the scale of competition constant.

Appendix

Noting (14), household ι 's expected utility (8) can be written as follows:

$$U(C_\iota, T) = E \int_T^\infty f(1 + \beta)u(C_\iota, N_\iota, y)^{1-\epsilon} e^{-\rho(t-T)} dt.$$

Because the household takes the proportion of duopoly industries, β , as given, it is equivalent to maximize

$$E \int_T^\infty u(C_\iota, N_\iota, y)^{1-\epsilon} e^{-\rho(t-T)} dt. \quad (46)$$

Technology τ_k changes randomly in each industry k . I denote:

- $\{s_{\iota kv}\}$ vector of $s_{\iota kv}$ for $k \in [0, 1]$ and $v \in \{0, 1, 2\}$,
- $\{s_{\iota(k \neq j)v}\}$ vector of $s_{\iota kv}$ for $k \in [0, 1]$, $k \neq j$ and $v \in \{0, 1, 2\}$,
- $\{\tau_k\}$ vector of τ_k for $k \in [0, 1]$,
- $\{\tau_{k \neq j}\}$ vector of τ_k for $k \in [0, 1]$ and $k \neq j$.

I denote variables depending on $\{\tau_k\}$ by superscript $\{\tau_k\}$. Thus, $C_\iota^{\{\tau_k\}}$ is household ι 's current consumption, $y^{\{\tau_k\}}$ current average consumption, $w^{\{\tau_k\}}$ the current wage, $P^{\{\tau_k\}}$ the current price and

$$u^{\{\tau_k\}} \doteq u(C_\iota^{\{\tau_k\}}, N_\iota^{\{\tau_k\}}, y^{\{\tau_k\}}). \quad (47)$$

I define the value functions:

$\Omega(\{s_{\iota kv}\}, \{\tau_k\})$ the value of receiving profits $s_{\iota kv}$ from all firms v in all industries k using current technology τ_k .

$\Omega(\Pi_{i_{j\kappa}}, 0, \{s_{i(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\})$ the value of receiving the profit $\Pi_{i_{j\kappa}}$ from firm κ in industry $j \notin \Theta$ using technology $\tau_j + 1$, but receiving no profits from the other firm that was a producer in that industry when technology τ_j was used, and receiving profits $s_{i(k \neq j)v}$ from all firms v in other industries $k \neq j$ with current technology τ_k .

$\Omega(\phi \Pi_{i_{j1}}, (1 - \phi) \Pi_{i_{j0}}, \{s_{i(k \neq j)v}\}, \{\tau_k\})$ the value of receiving profit $\phi \Pi_{i_{j1}}$ from firm 1 and profit $(1 - \phi) \Pi_{i_{j0}}$ from firm 2 in industry $j \in \Theta$, but profits $s_{i(k \neq j)v}$ from all firms v in the other industries $k \neq j$ with current technology τ_k .

Let $\Lambda_{j\kappa}$ be the arrival rate of innovations that change duopolist κ into a monopoly in industry $j \notin \Theta$. Each of these innovations increases the value of that duopolist by the amount

$$\Omega(\Pi_{i_{j\kappa}}, 0, \{s_{i(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\}) - \Omega(\{s_{i_{k\kappa}v}\}, \{\tau_k\}),$$

Let Γ_{j0} be the arrival rate of imitations that change outsider 0 into the second duopolist in industry $j \in \Theta$. Each of these imitations increases the value of that outsider by the amount

$$\Omega(\phi \Pi_{i_{j1}}, (1 - \phi) \Pi_{i_{j0}}, \{s_{i(k \neq j)v}\}, \{\tau_k\}) - \Omega(\{s_{i_{k0}v}\}, \{\tau_k\}).$$

Household ι maximizes its utility (46) by its labor supply N_ι and investment, $\{S_{i_{j0}}\}$ for $j \in \Theta$ and $\{S_{i_{j1}}, S_{i_{j2}}\}$ for $j \notin \Theta$, subject to (19), (20), (21) and (22), taking the macroeconomic variables w , y , P and $\{\Lambda_{j\kappa}, \Gamma_{j0}\}$ for all j and κ as given. The Bellman equation associated with the household's maximization is given by¹⁶

$$\rho \Omega(\{s_{i_{k\kappa}v}\}, \{\tau_k\}) = \max_{S_{i_{j0}} \geq 0 \text{ for all } j} \Xi_\iota \quad (48)$$

¹⁶Cf. Dixit and Pindyck (1994).

with

$$\begin{aligned}
\Xi_\iota &\doteq u(C_\iota^{\{\tau_k\}}, N_\iota^{\{\tau_k\}}, y^{\{\tau_k\}})^{1-\epsilon} \\
&+ \int_{j \in \Theta} \Gamma_{j0} \left[\Omega(\Pi i_{\iota j \kappa}, 0, \{s_{\iota(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\}) - \Omega(\{s_{\iota k v}\}, \{\tau_k\}) \right] dj \\
&+ \int_{j \notin \Theta} \sum_{\kappa=1,2} \Lambda_{j\kappa} \left[\Omega(\phi \Pi i_{\iota j 1}, (1-\phi) \Pi i_{\iota j 0}, \{s_{\iota(k \neq j)v}\}, \{\tau_k\}) \right. \\
&\quad \left. - \Omega(\{s_{\iota k v}\}, \{\tau_k\}) \right] dj. \tag{49}
\end{aligned}$$

The first-order condition corresponding to labor supply N_ι is given by

$$N_\iota = \arg \max_{N_\iota} u(C_\iota, N_\iota, y)^{1-\epsilon} = \arg \max_{N_\iota} u(C_\iota, N_\iota, y). \tag{50}$$

Because $\partial C_\iota / \partial S_{\iota j \kappa} = w/P$ by (21) and (22), then from (1), (7), (47) and (50) it follows that

$$\begin{aligned}
0 &= \frac{du^{\{\tau_k\}}}{dN_\iota^{\{\tau_k\}}} \Big|_{C_\iota^{\{\tau_k\}}=y^{\{\tau_k\}}} = \left[\frac{\partial u^{\{\tau_k\}}}{\partial C_\iota^{\{\tau_k\}}} \frac{w^{\{\tau_k\}}}{P^{\{\tau_k\}}} + \frac{\partial u^{\{\tau_k\}}}{\partial N_\iota^{\{\tau_k\}}} \right]_{C_\iota^{\{\tau_k\}}=y^{\{\tau_k\}}} \\
&= \frac{\partial u^{\{\tau_k\}}}{\partial C_\iota^{\{\tau_k\}}} \frac{w^{\{\tau_k\}} - \xi P^{\{\tau_k\}} y^{\{\tau_k\}}}{P^{\{\tau_k\}}} = \frac{\partial u^{\{\tau_k\}}}{\partial C_\iota^{\{\tau_k\}}} \frac{w^{\{\tau_k\}} - \xi}{P^{\{\tau_k\}}}, \\
w^{\{\tau_k\}} \Big|_{C_\iota^{\{\tau_k\}}=y^{\{\tau_k\}}} &= \xi = \text{constant}. \tag{51}
\end{aligned}$$

Because $\partial C_\iota / \partial S_{\iota j \kappa} = -1/P^{\{\tau_k\}}$ by (21) and (22), the first-order conditions corresponding to investment, $\{S_{\iota j 0}\}$ for $j \in \Theta$ and $\{S_{\iota j 1}, S_{\iota j 2}\}$ for $j \notin \Theta$, are

$$\begin{aligned}
&\Lambda_{j\kappa} \frac{d}{dS_{\iota j \kappa}} \left[\Omega(\Pi i_{\iota j \kappa}, 0, \{s_{\iota(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\}) - \Omega(\{s_{\iota k v}\}, \{\tau_k\}) \right] \\
&= (1-\epsilon) (u^{\{\tau_k\}})^{-\epsilon} \frac{\partial u^{\{\tau_k\}}}{\partial C_\iota^{\{\tau_k\}}} \frac{1}{P^{\{\tau_k\}}} \text{ for } j \notin \Theta \text{ and } \kappa \in \{1, 2\}, \tag{52}
\end{aligned}$$

$$\begin{aligned}
&\Gamma_{j0} \frac{d}{dS_{\iota j 0}} \left[\Omega(\phi \Pi i_{\iota j 1}, (1-\phi) \Pi i_{\iota j 0}, \{s_{\iota(k \neq j)v}\}, \{\tau_k\}) - \Omega(\{s_{\iota k v}\}, \{\tau_k\}) \right] \\
&= (1-\epsilon) (u^{\{\tau_k\}})^{-\epsilon} \frac{\partial u^{\{\tau_k\}}}{\partial C_\iota^{\{\tau_k\}}} \frac{1}{P^{\{\tau_k\}}} \text{ for } j \in \Theta. \tag{53}
\end{aligned}$$

I try the solution that for each household ι the propensity to consume, h_ι , and the subjective interest rate r_ι are independent of income A_ι . Since according to (22) income $A_\iota^{\{\tau_k\}}$ depends directly on variables $\{s_{\iota k v}^{\tau_k}\}$, I denote

$A_l^{\{\tau_k\}}(\{s_{lkv}^{\tau_k}\})$. Assuming that the propensity to consume, h_l , is invariant across technologies $\{\tau_k\}$, I obtain

$$P^{\{\tau_k\}}C_l^{\{\tau_k\}} = h_l A_l^{\{\tau_k\}}(\{s_{lkv}^{\tau_k}\}). \quad (54)$$

The share in the next innovator $\tau_j + 1$ is determined by investment under the present technology τ_j , $s_{lj\kappa}^{\tau_j+1} = \Pi i_{lj\kappa}^{\tau_j}$ for $j \notin \Theta$. The share in the next imitator is determined by investment under the same technology τ_j , $s_{lj\kappa}^{\tau_j} = (1 - \phi)\Pi i_{lj\kappa}^{\tau_j}$ for $j \in \Theta$. The value functions are then given by

$$\begin{aligned} \Omega(\{s_{lkv}\}, \{\tau_k\}) &= \frac{1}{r_l} (u^{\{\tau_k\}})^{1-\epsilon}, \\ \Omega(\Pi i_{lj\kappa}, 0, \{s_{l(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\}) &= \frac{1}{r_l} (u^{\tau_j+1, \{\tau_{k \neq j}\}})^{1-\epsilon}. \end{aligned} \quad (55)$$

From (22), (54) and (55) it follows that

$$\begin{aligned} A_l^{\{\tau_k\}}(\phi \Pi i_{lj1}, (1 - \phi)\Pi i_{lj0}, \{s_{l(k \neq j)v}^{\tau_k}\}) \Big|_{i_{lj0}=i_{lj1}} &= A_l^{\{\tau_k\}}(\{s_{lkv}^{\tau_k}\}), \\ C_l^{\{\tau_k\}}(\phi \Pi i_{lj1}, (1 - \phi)\Pi i_{lj0}, \{s_{l(k \neq j)v}^{\tau_k}\}) \Big|_{i_{lj0}=i_{lj1}} &= C_l^{\{\tau_k\}}(\{s_{lkv}^{\tau_k}\}), \\ \Omega(\phi \Pi i_{lj1}, (1 - \phi)\Pi i_{lj0}, \{s_{l(k \neq j)v}\}, \{\tau_k\}) \Big|_{i_{lj0}=i_{lj1}} &= \Omega(\{s_{lkv}\}, \{\tau_k\}). \end{aligned} \quad (56)$$

Given (55) and (56), one obtains

$$\frac{\partial \Omega(\{s_{lkv}\}, \{\tau_k\})}{\partial s_{lj}^{\tau_j}} = 0. \quad (57)$$

From (7), (19), (22), (54), (55), $s_{lj\kappa}^{\tau_j+1} = \Pi i_{lj\kappa}^{\tau_j}$ for $j \notin \Theta$ and $\kappa = 1, 2$, and $s_{lj2}^{\tau_j} = (1 - \phi)\Pi i_{lj0}^{\tau_j}$ for $j \in \Theta$ it follows that

$$\begin{aligned} \frac{\partial s_{lj\kappa}^{\tau_j+1}}{\partial i_{lj\kappa}^{\tau_j}} &= \Pi \text{ for } j \notin \Theta \text{ and } \kappa = 1, 2, & \frac{\partial s_{lj2}^{\tau_j}}{\partial i_{lj0}^{\tau_j}} &= (1 - \phi)\Pi \text{ for } j \in \Theta, \\ \frac{\partial A_l^{\tau_j+1, \{\tau_{k \neq j}\}}}{\partial s_{lj0}^{\tau_j+1}} &= \frac{\partial A_l^{\{\tau_k\}}}{\partial s_{lj0}^{\tau_j}} = 1, & \frac{\partial i_{lj\kappa}^{\tau_j}}{\partial s_{lj\kappa}^{\tau_j}} &= \frac{1}{w^{\{\tau_k\}} l_{j\kappa}^{\{\tau_k\}}} \text{ for all } j \text{ and } \kappa, \end{aligned}$$

$$\begin{aligned}
& \frac{d\Omega(\Pi i_{lj\kappa}, 0, \{s_{l(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\})}{dS_{lj\kappa}^{\tau_j}} \\
&= \frac{1 - \epsilon}{r_l} (u^{\tau_j+1, \{\tau_{k \neq j}\}})^{-\epsilon} \frac{\partial u^{\tau_j+1, \{\tau_{k \neq j}\}}}{\partial C_l^{\tau_j+1, \{\tau_{k \neq j}\}}} \\
& \quad \frac{\partial C_l^{\tau_j+1, \{\tau_{k \neq j}\}}}{\partial A_l^{\tau_j+1, \{\tau_{k \neq j}\}}} \frac{\partial A_l^{\tau_j+1, \{\tau_{k \neq j}\}}}{\partial s_{lj\kappa}^{\tau_j+1}} \frac{\partial s_{lj\kappa}^{\tau_j+1}}{\partial i_{lj\kappa}^{\tau_j}} \frac{\partial i_{lj\kappa}^{\tau_j}}{\partial S_{lj\kappa}^{\tau_j}} \\
& \quad \underbrace{\frac{\partial A_l^{\tau_j+1, \{\tau_{k \neq j}\}}}{h_l/P^{\tau_j+1, \{\tau_{k \neq j}\}}}}_{=1} \underbrace{\frac{\partial s_{lj\kappa}^{\tau_j+1}}{\partial i_{lj\kappa}^{\tau_j}}}_{=\Pi} \\
&= \frac{1 - \epsilon}{r_l} (u^{\tau_j+1, \{\tau_{k \neq j}\}})^{-\epsilon} \frac{\partial u^{\tau_j+1, \{\tau_{k \neq j}\}}}{\partial C_l^{\tau_j+1, \{\tau_{k \neq j}\}}} \frac{\Pi h_l}{P^{\tau_j+1, \{\tau_{k \neq j}\}}} \frac{\partial i_{lj\kappa}^{\tau_j}}{\partial S_{lj\kappa}^{\tau_j}} \\
&= \frac{1 - \epsilon}{r_l} (u^{\tau_j+1, \{\tau_{k \neq j}\}})^{-\epsilon} \frac{\partial u^{\tau_j+1, \{\tau_{k \neq j}\}}}{\partial C_l^{\tau_j+1, \{\tau_{k \neq j}\}}} \frac{\Pi h_l}{w^{\{\tau_k\}} P^{\tau_j+1, \{\tau_{k \neq j}\}} l_{j\kappa}^{\{\tau_k\}}} \quad \text{for } j \notin \Theta, \quad (58)
\end{aligned}$$

$$\begin{aligned}
& \frac{d\Omega(\phi s_{lj1}, (1 - \phi)\Pi i_{lj2}, \{s_{l(k \neq j)v}\}, \{\tau_k\})}{dS_{lj0}^{\tau_j}} \\
&= \frac{1 - \epsilon}{r_l} (u^{\{\tau_k\}})^{-\epsilon} \frac{\partial u^{\{\tau_k\}}}{\partial C_l^{\{\tau_k\}}} \underbrace{\frac{\partial C_l^{\{\tau_k\}}}{\partial A_l^{\{\tau_k\}}}}_{=h_l/P^{\{\tau_k\}}} \underbrace{\frac{\partial A_l^{\{\tau_k\}}}{s_{lj0}^{\tau_j}}}_{=1} \underbrace{\frac{s_{lj0}^{\tau_j}}{\partial i_{lj0}^{\tau_j}}}_{=(1-\phi)\Pi} \frac{\partial i_{lj0}^{\tau_j}}{\partial S_{lj0}^{\tau_j}} \\
&= \frac{1 - \epsilon}{r_l} (u^{\{\tau_k\}})^{-\epsilon} \frac{\partial u^{\{\tau_k\}}}{\partial C_l^{\{\tau_k\}}} \frac{(1 - \phi)\Pi h_l}{P^{\{\tau_k\}}} \frac{\partial i_{lj0}^{\tau_j}}{\partial S_{lj0}^{\tau_j}} \\
&= \frac{1 - \epsilon}{r_l} (u^{\{\tau_k\}})^{-\epsilon} \frac{\partial u^{\{\tau_k\}}}{\partial C_l^{\{\tau_k\}}} \frac{(1 - \phi)\Pi h_l}{w^{\{\tau_k\}} P^{\{\tau_k\}} l_{j0}^{\{\tau_k\}}} \quad \text{for } j \in \Theta. \quad (59)
\end{aligned}$$

I focus on a stationary equilibrium where the growth rate g and the allocation of labor, $(l_{j\kappa}, x, N_l)$, are invariant across technologies. Because there is symmetry throughout all industries $j \in [0, 1]$ on one hand and throughout all households $j \in [0, 1]$ on the other hand, from (1), (3), (7), (13), (16) and (51) it follows that

$$\begin{aligned}
i_{lj0} &= i_{lj1}, \quad l_{j\kappa}^{\{\tau_k\}} = l_{j\kappa}, \quad x_j^{\{\tau_k\}} = x = N - l, \quad C_l^{\{\tau_k\}} = y^{\{\tau_k\}}, \\
\frac{u_l^{\{\tau_k\}}}{u_l^{\tau_j+1, \{\tau_{k \neq j}\}}} &= \frac{C_l^{\{\tau_k\}}}{C_l^{\tau_j+1, \{\tau_{k \neq j}\}}} = \frac{P^{\tau_j+1, \{\tau_{k \neq j}\}}}{P^{\{\tau_k\}}} = \frac{y^{\{\tau_k\}}}{y^{\tau_j+1, \{\tau_{k \neq j}\}}} = \frac{B^{\{\tau_k\}}}{B^{\tau_j+1, \{\tau_{k \neq j}\}}} = \frac{1}{\mu}, \\
w &= \text{constant}, \quad x = 1/(\mu w) = \text{constant}. \quad (60)
\end{aligned}$$

Inserting (17), (49), (55), (56) and (60) into (48) yields

$$0 = \left[\rho + \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj + \int_{j \in \Theta} \Gamma_{j0} dj \right] \Omega(\{s_{lkv}\}, \{\tau_k\})$$

$$\begin{aligned}
& - (u^{\{\tau_k\}})^{1-\epsilon} - \int_{j \notin \Theta} \sum_{\kappa=1,2} \Lambda_{j\kappa} \Omega(\Pi_{i_{j\kappa}}, 0, \{s_{i(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\}) dj \\
& - \int_{j \in \Theta} \Gamma_{j0} \Omega(\phi \Pi_{i_{j1}}, (1-\phi) \Pi_{i_{j0}}, \{s_{i(k \neq j)v}\}, \{\tau_k\}) dj \\
= & \left[\rho + \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj \right] \frac{1}{r_\iota} (u^{\{\tau_k\}})^{1-\epsilon} - (u^{\{\tau_k\}})^{1-\epsilon} \\
& - \int_{j \notin \Theta} \frac{\Lambda_{j1} + \Lambda_{j2}}{r_\iota} (u^{\tau_j+1\{\tau_{k \neq j}\}})^{1-\epsilon} dj \\
= & \frac{1}{r_\iota} (u^{\{\tau_k\}})^{1-\epsilon} \left[\rho + (1 - \mu^{1-\epsilon}) \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj - r_\iota \right] \\
= & \frac{1}{r_\iota} (u^{\{\tau_k\}})^{1-\epsilon} \left[\rho + \frac{1 - \mu^{1-\epsilon}}{\ln \mu} g - r_\iota \right].
\end{aligned}$$

This equation is equivalent to

$$r_\iota = \rho + \frac{1 - \mu^{1-\epsilon}}{\ln \mu} g. \quad (61)$$

Because there is symmetry throughout all households ι , their propensity to consume is equal, $h_\iota = h$. This, (5), (18), (21) and (54) yield

$$\begin{aligned}
wl &= w \int_{j \in \Theta} l_{j0} dj + w \int_{j \notin \Theta} (l_{j1} + l_{j2}) dj = w \int_{j \in \Theta} l_{j0} dj + \int_{j \notin \Theta} (l_{j1} + l_{j2}) dj \\
&= \int_0^1 \left[\int_{j \in \Theta} S_{ij0} dj + \int_{j \notin \Theta} (S_{ij1} + S_{ij2}) dj \right] d\iota = \int_0^1 (A_\iota - PC_\iota) d\iota \\
&= (1-h) \int_0^1 A_\iota d\iota = (1-h)(1+wl)
\end{aligned}$$

and

$$h_\iota = h = (1+wl)^{-1}. \quad (62)$$

Because there is perfect symmetry throughout all firms inside the sets of innovative and imitative industries, there are η and ψ so that

$$l_{j\ell} = \eta \text{ for } j \notin \Theta \text{ and } \ell = 1, 2, \quad l_{j0} = \psi \text{ for } j \in \Theta. \quad (63)$$

Given this and (14), the production function (10) changes into

$$\Lambda_{j\kappa} = \lambda l_{j\kappa}^{1-\delta} \left(\int_{k \notin \Theta} \sum_{\ell=1}^2 l_{k\ell} dk \right)^\delta = (2\beta)^\delta \lambda \eta. \quad (64)$$

Inserting (7), (11), (57), (58), (59), (60), (62), (63) and (64) into (52) and (53) yields

$$\begin{aligned}
& \mu^{1-\epsilon} \frac{(2\beta)^\delta \lambda}{r_l} (u^{\{\tau_k\}})^{-\epsilon} \frac{\partial u^{\{\tau_k\}}}{\partial C_l^{\{\tau_k\}}} \frac{\Pi h}{w P^{\{\tau_k\}}} = \mu^{1-\epsilon} \frac{\Lambda_{j\kappa}}{r_l} (u^{\{\tau_k\}})^{-\epsilon} \frac{\partial u^{\{\tau_k\}}}{\partial C_l^{\{\tau_k\}}} \frac{\Pi h_l}{w P^{\{\tau_k\}} \eta} \\
& = \frac{\Lambda_{j\kappa}}{r_l} (\mu u^{\{\tau_k\}})^{-\epsilon} \frac{\partial u^{\{\tau_k\}}}{\partial C_l^{\{\tau_k\}}} \frac{\Pi h_l}{w (P^{\{\tau_k\}}/\mu) l_{j\kappa}} \\
& = \frac{\Lambda_{j\kappa}}{r_l} (u^{\tau_j+1, \{\tau_{k \neq j}\}})^{-\epsilon} \frac{\partial u^{\tau_j+1, \{\tau_{k \neq j}\}}}{\partial C_l^{\tau_j+1, \{\tau_{k \neq j}\}}} \frac{\Pi h_l}{w^{\{\tau_k\}} P^{\tau_j+1, \{\tau_{k \neq j}\}} l_{j\kappa}^{\{\tau_k\}}} \\
& = \frac{\Lambda_{j\kappa}}{1-\epsilon} \frac{d\Omega(\Pi i_{lj\kappa}, 0, \{s_{l(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\})}{dS_{lj\kappa}^{\tau_j}} = (u^{\{\tau_k\}})^{-\epsilon} \frac{\partial u^{\{\tau_k\}}}{\partial C_l^{\{\tau_k\}}} \frac{1}{P^{\{\tau_k\}}}
\end{aligned} \tag{65}$$

for $j \notin \Theta$ and $\kappa \in \{1, 2\}$,

$$\begin{aligned}
& \frac{\lambda/a}{r_l} (u^{\{\tau_k\}})^{-\epsilon} \frac{\partial u^{\{\tau_k\}}}{\partial C_l^{\{\tau_k\}}} \frac{(1-\phi)\Pi h}{w P^{\{\tau_k\}}} = \frac{\Gamma_{j0}}{r_l} (u^{\{\tau_k\}})^{-\epsilon} \frac{\partial u^{\{\tau_k\}}}{\partial C_l^{\{\tau_k\}}} \frac{(1-\phi)\Pi h_l}{w^{\{\tau_k\}} P^{\{\tau_k\}} l_{j0}^{\{\tau_k\}}} \\
& = \frac{\Gamma_{j0}}{1-\epsilon} \frac{d\Omega(\phi s_{lj1}, (1-\phi)\Pi i_{lj2}, \{s_{l(k \neq j)v}\}, \{\tau_k\})}{dS_{lj0}^{\tau_j}} = (u^{\{\tau_k\}})^{-\epsilon} \frac{\partial u^{\{\tau_k\}}}{\partial C_l^{\{\tau_k\}}} \frac{1}{P^{\{\tau_k\}}}
\end{aligned} \tag{66}$$

for $j \in \Theta$.

Given (65) and (66), one obtains

$$\beta = \frac{1}{2} \left(\frac{1-\phi}{\mu^{1-\epsilon} a} \right)^{1/\delta} \tag{67}$$

Equations (13), (14), (15), (17), (61), (62), (64) and (65) yield

$$\begin{aligned}
l &= \int_{j \notin \Theta} (l_{j1} + l_{j2}) dj + \int_{j \in \Theta} l_{j0} dj = 2\eta \int_{j \notin \Theta} dj + \psi \int_{j \in \Theta} dj = (1-\beta)\psi + 2\beta\eta, \\
\psi &= (l - 2\beta\eta)/(1-\beta),
\end{aligned} \tag{68}$$

$$g = (\ln \mu) \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj = (2 \ln \mu) \beta \Lambda_{j\kappa} = (2^{1+\delta} \lambda \ln \mu) \beta^{1+\delta} \eta, \tag{69}$$

$$\left(\rho + \frac{1-\mu^{1-\epsilon}}{\ln \mu} g \right) \frac{\mu^{\epsilon-1}}{(2\beta)^\delta \lambda} = \frac{r_l \mu^{\epsilon-1}}{(2\beta)^\delta \lambda} = \frac{h\Pi}{w} = \frac{\Pi}{(1+wl)w}. \tag{70}$$

Relations (60), (67), (69) and (70) yield (24)-(26). \square

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