Directed Technical Change, Scale Effects and Industrial Structure

Pedro Mazeda Gil[*], Óscar Afonso[†], Paulo Brito[‡]

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This paper studies a specific dimension of the industrial structure - the distribution of firms and production across sectors - by building an endogenous-growth model of directed technical change that merges the expanding-variety with the quality-ladders mechanism. By recognising the complementarity between those two mechanisms within a full lab-equipment specification, the model generates a specific set of results, namely with respect to the steady-state industrial structure of countries with different levels of relative labour endowment, and the relationship between structure, long-term aggregate growth and R&D intensity.

Keywords: endogenous growth, industrial structure, directed technical change, scale effects, labour endowment

JEL Classification: O41, D43, L11, L16

1. Introduction

The identification of the determinants of industrial structure follow a long-standing tradition in industrial organisation (IO). In recognising the endogenous character of (many of) those determinants, the literature has frequently emphasised the interrelation between industrial structure, and technology and innovative activity (e.g., Dasgupta and Stiglitz, 1980; and Sutton, 1998).

Recently, a strand of the literature studies the interplay between long-term aggregate growth, innovative activity and factors usually analysed in IO domain. Whereas some

[*]Faculty of Economics, University of Porto, and CEF.UP. Corresponding author: please email to pgil@fep.up.pt or address to Rua Dr Roberto Frias, 4200-464, Porto, Portugal.

[†]Faculty of Economics, University of Porto, and CEF.UP

[‡]School of Economics and Management, Technical University of Lisbon, and UECE

papers explore the interdependence between structure and growth, by focusing on the strategic interaction of firms in an oligopolistic framework (e.g., van de Klundert and Smulders, 1997; Peretto, 1999; Aghion, Bloom, Blundell, Griffith, and Howitt, 2005), others emphasise the role of a specific factor within a monopolistic-competition setting - e.g., the number of firms (Peretto and Smulders, 2002), average firm size (e.g., Peretto, 1998) or firm size distribution (e.g., Thompson, 2001; Klette and Kortum, 2004).

This paper relates closely to the latter set of papers, while connecting endogenous directed (or skill-biased) technical change literature (e.g, Acemoglu, 1998; Kiley, 1999; Acemoglu and Zilibotti, 2001) with the study of long-term firm dynamics and industrial structure. In particular, it combines vertical and horizontal R&D and scale effects related with factor endowment to analyse the industrial structure, as characterised by the number of (monopolistic) firms, production and average firm size across sectors. "Sector" herein represents a group of firms producing the same type of labour-complementary intermediate goods: we follow the literature and dichotomise between unskilled-labour complementary-technology sector and skilled-labour complementary-technology sector.

Figure 1, Appendix A, illustrates data with respect to the number of firms, production, average firm size and R&D intensity in high-tech vis-à-vis low-tech sectors, as defined by the OECD, across some developed countries. We take high-tech and low-tech sectors as the empirical counterpart of the unskilled-labour and skilled-labour complementary-technology sectors in our model (e.g., Cozzi and Impullitti, 2008). The data suggests a considerable variability of industrial structures across countries, although some regularities can be pointed out: (i) the number of firms and total production are smaller in the high-tech than in the low-tech sectors (relative number of firms and relative production below unity); (ii) average firm size and R&D intensity are larger in the former; (iii) relative production tends to be positively related to the relative number of firms; (iv) relative R&D intensity tends to be negatively related to relative firm size.

There are several studies analysing the link between the distribution of economic activity across industries and growth, either empirically (e.g., Fagerberg, 2000) or theoretically (e.g., Ngai and Pissarides, 2007; Bonatti and Felice, 2008; Acemoglu and Guerrieri, 2008). They derive the implications of different sectoral total factor productivity (TFP) growth rates, suggesting that countries specialized in "technologically progressive" industries (high TFP growth) enjoy higher growth rates.By analysing the same particular dimension of industrial structure, our paper is related with this literature. However, by building on a mechanism of endogenous directed technical change, it is substantially different. It predicts constant TFP growth rates across sectors along the balanced growth path (see Acemoglu and Zilibotti, 2001). Thus, concerning the link between growth and industrial structure, our results are set in *quantitative* terms, i.e., how many firms and how much production are allocated to each sector vis-à-vis the others, and not in *qualitative* terms, i.e., in which specific sector is economic activity concentrated.

Our basic setup is an endogenous-growth model that merges the expanding-variety with the quality-ladders mechanism, in line with, e.g., Dinopoulos and Thompson (1998) and Howitt (1999). The main motivation behind these early models is the removal of scale effects of population growth within a knowledge-driven R&D specification. In consequence, they predict that the steady-state flow of new goods grows at the same (exogenous) rate as the population. We consider a full lab-equipment specification, i.e., the input to R&D activities *and* to differentiated-goods production is the homogeneous final good (e.g., Segerstrom and Zolnierek, 1999),[1] which, despite the degree of scale-effect removal, allows for a fully endogenous expanding-variety mechanism, such that the flow of new goods is independent of population growth.

Acemoglu (2002) observes that the consideration of horizontal or vertical R&D is redundant for directed technical change. However, in line with Peretto and Connolly (2007), we accommodate the view that vertical R&D allows for growth "unconstrained by endowments", while horizontal R&D permits an explicit link between aggregate and industry-level variables (number of firms, firm size and entry rate). By recognising the complementarity between those R&D types within a full lab-equipment specification, we find a specific set of results, namely regarding the industrial structure of countries with different relative labour endowment (the ratio between skilled and unskilled labour measured in efficiency units) and distinct degrees of scale effects.

The quality-ladders mechanism provides the accumulation of non-physical capital (technological knowledge), whilst the expanding-variety mechanism offers the flow of new firms (new product lines). Average firm size is thus measured as technological-knowledge stock per firm, which, however, relates closely to production (sales) per firm or financial assets per firm. On the other hand, by identifying the feedbacks between structure, innovative activity and growth performance, our general-equilibrium framework makes explicit the endogenous determination of industrial structure, in line with the literature that develops the "Schumpeterian hypotheses". In particular, a subset of the technology parameters that determine R&D activity and aggregate growth rate simultaneously affect the industrial structure and long-term industry dynamics.

In lab-equipment models, the scale effects are connected with the size of profits that, in each period, accrue to the incumbent; a larger market expands the incumbent's profits and, thus, the incentives to allocate resources to R&D, thereby increasing the aggregate growth rate.[2]In our model, an increase in market scale also dilutes the impact of R&D outlays on innovation probability (e.g., Barro and Sala-i-Martin, 2004), since coordination, organisational, informational, marketing and transportation costs (e.g., Dinopoulos and Thompson, 1999) and rental protection actions by incumbents (e.g., Sener, 2008) (positively) related to market size make the introduction of new intermediate goods and the replacement of old ones increasingly difficult as the market grows. However, depending on the effectiveness of the referred costs and actions, these may partial, totally or over counterbalance the benefits of scale to innovative activity.

Accordingly, we allow for varying degrees of scale effects and, by focusing on the steady state, find that, as the degree of scale effects changes, the industrial structure associated to a given level of relative labour endowment may differ significantly. Likewise, the relationship between industrial structure and both R&D intensity (a version of the

[1]Using Rivera-Batiz and Romer (1991)'s terminology, the assumption that final good is the R&D input means that one adopts the "lab-equipment" version of R&D, instead of the "knowledge-driven" specification, in which labour is the only input.

[2]The basic theory of industrial structure in IO literature suggests that concentration is determined by economies of scale relative to the market size. In our model, the role of market size is quite different.

"Schumpeterian hypothesis") and long-term aggregate growth may depend on the given combination between scale effects and relative labour endowment. By generating a wide set of results, our model hence provides a possible economic mechanism to explain the empirical facts in Figure 1.

The remainder of the paper has the following structure. In Section 2, we give an overview of the model of directed technological change with quality ladders and horizontal entry. Section 3 presents the general equilibrium and analyses the steady-state properties of the model. In Section 4, we detail the comparative steady-state results, namely with respect to the degree of scale effects and the level of relative labour endowment. Section 5 gives some concluding notes.

2. The model

2.1. Production and price decisions

This is a model of directed technological change with quality ladders and horizontal entry, built into a dynamic general equilibrium setup of a closed economy where there is a single competitively-produced final good, Y, that can be used in consumption, C, production of intermediate goods, X, and R&D activities, R. The final good is produced by a continuum of firms, indexed by $n \in [0,1]$, each using labour and a continuum of intermediate inputs indexed by $\omega \in [0, N(t)]$. Aggregate output (in terms of the composite final good) is defined as $Y(t) = \int_o^1 P(n,t)Y(n,t)dn = exp\left[\int_o^1 lnY(n,t)dn\right]$. We treat the composite final good as numeraire and normalise its price to unity in each period, such that $exp\left[\int_0^1 lnP(n)dn\right] = P_Y = 1$. For concreteness, assume that, as in Acemoglu and Zilibotti (2001), there are two types of intermediate goods - one is unskilled-labour complementary and the other is skilled-labour complementary -, such that each intermediate good can be used either only by unskilled workers or by skilled workers. The two technological groups enter the final-good production function as specified below

$$Y(n,t) = A\left[\int_0^{N_L(t)} \left(\lambda^{j_L(\omega,t)} \cdot x_L(n,\omega,t)\right)^{1-\alpha} d\omega\right] [(1-n)\cdot l \cdot L(n)]^\alpha + \\ +A\left[\int_0^{N_H(t)} \left(\lambda^{j_H(\omega,t)} \cdot x_H(n,\omega,t)\right)^{1-\alpha} d\omega\right] [n \cdot h \cdot H(n)]^\alpha \quad (1)$$

where $A > 0$ is a given scale parameter; L and H are unskilled and skilled labour, respectively; and, as in Afonso (2006), $x_m(n,\omega,t)$ is the amount used by final-good firm n of the m-complementary ($m = L, H$) intermediate good ω, weighted by its quality level $\lambda^{j_m(\omega,t)}$. It is implicit in (1) that only the highest grade of each $\omega \in [0, N_L(t)] \cup [0, N_H(t)]$ are actually produced and used in equilibrium, meaning $x_m(j,\omega,t) = x_m(\omega,t)$;[3] thus, $N_m(t) > 0$ is the measure of how many different m-complementary intermediate goods ω exist at time t, such that $N_L(t) + N_H(t) = N(t)$. The contribution of the specific labour

[3]Henceforth, we use explicitly all arguments (j,ω,t) when they are useful for convenience of exposition.

inputs are condensed by the terms with exponent $0 < \alpha < 1$. These terms include two corrective factors for productivity differentials. An *absolute* productivity advantage of high over low-skilled labour is captured by $h > l > 1$. A *relative* productivity advantage of either type is obtained by n and $(1-n)$: high-skilled labour is relatively more productive in final goods indexed by larger n, and vice-versa. As explained below, at each t there is a competitive equilibrium threshold final good $\bar{n}$, endogenously determined, where the switch from one technology to the other becomes advantageous, so that each final good is produced exclusively with one technology, either high or low.

For the time being, we take N_m as given and follow Acemoglu and Zilibotti (2001)'s derivations from close. Each firm n in the final-good sector seeks to maximise profit by solving

$$\max_{\{x(n,\omega,t),\omega\in[0,N_L(t)]\cup[0,N_H(t)]\},L(n),H(n)} P(n,t)\cdot Y(n,t) - w_L(t)\cdot L(n) - w_H(t)\cdot H(n) -$$

$$-\int_0^{N_L(t)} p_L(\omega,t)\cdot x_L(n,\omega,t)d\omega - \int_0^{N_H(t)} p_H(\omega,t)\cdot x_H(n,\omega,t)d\omega \qquad (2)$$

where $P(n,t)$, the price of the final good sold by n, $p_m(\omega,t)$, the price of m-complementary ω, and $w_m(t)$, the labour wage of m at t are taken as given by n. The solution to this problem implies that the demand for ω is

$$\begin{aligned} x_L(n,\omega,t) &= (1-n)\cdot l\cdot L(n)\cdot\left[\tfrac{A\cdot P(n,t)\cdot(1-\alpha)}{p_L(\omega,t)}\right]^{\frac{1}{\alpha}}\lambda^{j(\omega,t)\left(\frac{1-\alpha}{\alpha}\right)} \\ x_H(n,\omega,t) &= n\cdot h\cdot H(n)\cdot\left[\tfrac{A\cdot P(n,t)\cdot(1-\alpha)}{p_H(\omega,t)}\right]^{\frac{1}{\alpha}}\lambda^{j(\omega,t)\left(\frac{1-\alpha}{\alpha}\right)} \end{aligned} \qquad (3)$$

Given (1) and (3), final-good output is

$$Y(n,t) = A^{\frac{1}{\alpha}}\cdot P(n,t)^{\frac{1-\alpha}{\alpha}}\cdot\left(\frac{1-\alpha}{p(\omega,t)}\right)^{\frac{1-\alpha}{\alpha}}\cdot[(1-n)\cdot l\cdot L(n)\cdot Q_L(t) + n\cdot h\cdot H(n)\cdot Q_H(t)] \qquad (4)$$

where

$$Q_m(t) = \int_0^{N_m(t)} \lambda^{j_m(\omega,t)\left(\frac{1-\alpha}{\alpha}\right)}d\omega \qquad (5)$$

is the intermediate-input aggregate quality index, or the *technological-knowledge stock*, for the technology group $m = L, H$. In (4), we have considered $p_m(\omega,t)\equiv p(\omega,t)$, as shown below.

The intermediate good is nondurable and entails a unit marginal cost of production, measured in terms of final-good output Y. Since there is a continuum of intermediate goods, one can assume that firms are atomistic and take as given the price of final output (numeraire). Monopolistic competition, therefore, prevails and firms face isoelastic demand curves $X_m(\omega) = \int_0^1 x_m(n,\omega)dn$ (see (3)). Leading-edge m-complementary ω producers choose their prices $p_m(\omega,t)$ to solve the profit maximization problem

$$\max_{p_m(\omega,t)} (p_m(\omega,t)-1)\cdot X_m(\omega,t) \tag{6}$$

Solving the first-order condition yields the optimal intermediate-good price

$$p_m(\omega,t) \equiv p = \frac{1}{1-\alpha} \tag{7}$$

which, since $0 < \alpha < 1$, is the usual monopoly price markup, constant over time and across industries.

Each m-complementary intermediate good is potentially sold to every $n \in [0,1]$. However, it can be shown that there is a threshold $\bar{n}$ that divides the mass of final-good firms into two groups, such that $0 \leq n \leq \bar{n}$ employs L-complementary x and $\bar{n} < n \leq 1$ employs H-complementary x (see Acemoglu and Zilibotti, 1999). Given (7), we can then write the final-good output as

$$Y(n,t) = \begin{cases} A^{\frac{1}{\alpha}} P(n,t)^{\frac{1-\alpha}{\alpha}} \cdot (1-\alpha)^{\frac{2(1-\alpha)}{\alpha}} \cdot (1-n) \cdot l \cdot L(n) \cdot Q_L(t) & , 0 \leq n \leq \bar{n} \\ A^{\frac{1}{\alpha}} P(n,t)^{\frac{1-\alpha}{\alpha}} \cdot (1-\alpha)^{\frac{2(1-\alpha)}{\alpha}} \cdot n \cdot h \cdot H(n) \cdot Q_H(t) & , \bar{n} \leq n \leq 1 \end{cases} \tag{8}$$

The equalisation of the marginal value product $\frac{\partial(P(n)Y(n))}{\partial m(n)}$ over $n \in [0,\bar{n}]$ (for L) and over $n \in [\bar{n},1]$ (for H), in equilibrium, implies $P(n,t)^{\frac{1}{\alpha}} \cdot (1-n)$ and $P(n,t)^{\frac{1}{\alpha}} \cdot n$ must be constant over $n \in [0,\bar{n}]$ and $n \in [\bar{n},1]$, respectively. Thus, define the final-good price indeces

$$\begin{aligned} P_L(t)^{\frac{1}{\alpha}} &= P(n,t)^{\frac{1}{\alpha}} \cdot (1-n) \\ P_H(t)^{\frac{1}{\alpha}} &= P(n,t)^{\frac{1}{\alpha}} \cdot n \end{aligned} \tag{9}$$

constant over $n \in [0,\bar{n}]$ and $n \in [\bar{n},1]$, respectively. Also, given Cobb-Douglas technology, expenditures across final goods are equalised (i.e., $P(n)Y(n)$ is constant over n), which implies $L(n)$ and $H(n)$ constant over $n \in [0,\bar{n}]$ and $n \in [\bar{n},1]$, respectively. Thus, we have the following labour-market clearing conditions

$$\begin{aligned} \int_0^{\bar{n}} L(n)dn = L \Leftrightarrow L(n) = \tfrac{L}{\bar{n}} \\ \int_{\bar{n}}^1 H(n)dn = H \Leftrightarrow H(n) = \tfrac{H}{1-\bar{n}} \end{aligned} \tag{10}$$

On the other hand, by solving (9) in order to $P(n)$ and noting that, in sector $\bar{n}$, a firm that uses unskilled workers and a firm that uses skilled workers should break even, we have, for $n = \bar{n}$,[4]

$$\frac{P_H}{P_L} = \left(\frac{\bar{n}}{1-\bar{n}}\right)^{\alpha} \tag{11}$$

Again, since $P(n) \cdot Y(n)$ is constant over n, then $P_H \cdot Y(1) = P_L \cdot Y(0)$. Given $Y(1)$, $Y(0)$, (10) and (11), we get

[4]Henceforth, we suppress time indexes when this causes no confusion.

$$\frac{P_H}{P_L} = \left(\frac{lLQ_L}{hHQ_H}\right)^{\frac{\alpha}{2}} \tag{12}$$

and

$$\bar{n} = \left[1 + \left(\frac{hHQ_H}{lLQ_L}\right)^{\frac{1}{2}}\right]^{-1} \tag{13}$$

Having in mind the numeraire rule $exp\left[\int_0^1 lnP(n)dn\right] = P_Y = 1$, some algebraic manipulation of the latter together with (11) and (13) yields

$$\begin{aligned} P_L = e^{-\alpha}\bar{n}^{-\alpha} = e^{-\alpha}\left[1 + \left(\tfrac{hHQ_H}{lLQ_L}\right)^{\frac{1}{2}}\right]^{\alpha} \\ P_H = e^{-\alpha}(1-\bar{n})^{-\alpha} = e^{-\alpha}\left[1 + \left(\tfrac{hHQ_H}{lLQ_L}\right)^{-\frac{1}{2}}\right]^{\alpha} \end{aligned} \tag{14}$$

With respect to aggregate quantities, see that, given (3), (7), (9) and the definition of $X_m(\omega)$, the optimal intermediate-good production for ω is

$$\begin{aligned} X_L(\omega) = \int_0^{\bar{n}} x_L(n,\omega)dn = A^{\frac{1}{\alpha}} \cdot (1-\alpha)^{\frac{2}{\alpha}} \cdot P_L^{\frac{1}{\alpha}} \cdot l \cdot L \cdot \lambda^{j_L(\omega)\left(\frac{1-\alpha}{\alpha}\right)} \\ X_H(\omega) = \int_{\bar{n}}^1 x_H(n,\omega)dn = A^{\frac{1}{\alpha}} \cdot (1-\alpha)^{\frac{2}{\alpha}} \cdot P_H^{\frac{1}{\alpha}} \cdot h \cdot H \cdot \lambda^{j_H(\omega)\left(\frac{1-\alpha}{\alpha}\right)} \end{aligned} \tag{15}$$

Given (15), the optimal profit accrued by intermediate-good sector monopolists is

$$\begin{aligned} \pi_L(\omega) = \bar{\pi}_L \cdot P_L^{\frac{1}{\alpha}} \cdot \lambda^{j_L(\omega)\left(\frac{1-\alpha}{\alpha}\right)} \\ \pi_H(\omega) = \bar{\pi}_H \cdot P_H^{\frac{1}{\alpha}} \cdot \lambda^{j_H(\omega)\left(\frac{1-\alpha}{\alpha}\right)} \end{aligned} \tag{16}$$

where $\bar{\pi}_L \equiv lLA^{\frac{1}{\alpha}}\left(\frac{\alpha}{1-\alpha}\right)(1-\alpha)^{\frac{2}{\alpha}}$ and $\bar{\pi}_H \equiv hHA^{\frac{1}{\alpha}}\left(\frac{\alpha}{1-\alpha}\right)(1-\alpha)^{\frac{2}{\alpha}}$ are positive constants. Observe that $\pi_m(\omega)$ jumps every time quality is upgraded in m-complementary ω, but changes continuously with P_m, as the dynamics of the latter reflects the continuum of jumps at the aggregate of industries.[5] Also from (15) and the definition of X, we see that the total intermediate-good optimal production is

$$\begin{aligned} X \equiv X_L + X_H \equiv \int_0^{N_L} X_L(\omega)d\omega + \int_0^{N_H} X_H(\omega)d\omega = \\ = A^{\frac{1}{\alpha}} \cdot (1-\alpha)^{\frac{2}{\alpha}} \cdot \left[P_L^{\frac{1}{\alpha}} \cdot l \cdot L \cdot Q_L + P_H^{\frac{1}{\alpha}} \cdot h \cdot H \cdot Q_H\right] \end{aligned} \tag{17}$$

From (8), (9)- and the definition of Y, we find the total final-good optimal production

[5] The uncertainty associated with vertical R&D at the industry level creates jumpiness in microeconomic outcomes. However, as we see in Subsection 2.2, below, as the probabilities of successful R&D across industries are independent and there is a continuum of industries, this jumpiness is not transmitted to macroeconomic variables, which hence can be treated as following a continuous non-stochastic time path.

$$Y \equiv Y_L + Y_H \equiv \int_0^{\bar{n}} P(n)Y(n)dn + \int_{\bar{n}}^1 P(n)Y(n)dn =$$

$$= A^{\frac{1}{\alpha}} \cdot (1-\alpha)^{\frac{2(1-\alpha)}{\alpha}} \cdot \left[P_L^{\frac{1}{\alpha}} \cdot l \cdot L \cdot Q_L + P_H^{\frac{1}{\alpha}} \cdot h \cdot H \cdot Q_H \right] \tag{18}$$

Finally, we can use (18), together with (14), to compute the skill premium as

$$\frac{w_H}{w_L} = \frac{h}{l} \left(\frac{hH}{lL} \right)^{-\frac{1}{2}} \left(\frac{Q_H}{Q_L} \right)^{\frac{1}{2}} \tag{19}$$

where $w_L = \frac{\partial Y}{\partial L}$ and $w_H = \frac{\partial Y}{\partial H}$.

2.2. R&D decisions

In line with Gil, Brito, and Afonso (2008), we assume that, in the intermediate-good sector, firms can devote resources to R&D either to create a new product line (a new industry) or, within an existing industry ω, to improve the quality of its good.

Vertical R&D free-entry and dynamic arbitrage conditions

As in the standard model of quality ladders, firms decide over their optimal vertical-R&D level, which constitutes the search for new designs (blueprints) that lead to a higher quality of existing intermediate goods. Each new design is granted a patent, meaning that a successful researcher retains exclusive rights over the use of his/her improved intermediate good. In each industry only (potential) entrants can do R&D and innovation arrival follows a Poisson process. There is free entry into each vertical R&D race and each entrant possesses the same R&D technology. Since there is perfect competition among entrants, the individual contribution of any particular entrant to the aggregate R&D expenditures of all entrants is negligible.[6]

Let $I_m^i(j, \omega, t)$ denote the instantaneous probability of R&D success by potential entrant i in m-complementary industry ω when the highest quality is j (I is also interpreted as the vertical innovation rate). This probability is independently distributed across firms, industries and over time, and depends on the flow of resources $R_{vm}^i(j, \omega, t)$ devoted to R&D by entrants in each m-complementary ω at t (measured in units of final-good output Y). As in, e.g., Barro and Sala-i-Martin (2004, ch. 7), we assume that each entrant's instantaneous probability of R&D success is given by a relation exhibiting constant returns in R&D expenditures, $I_m^i(j, \omega, t) = R_{vm}^i(j, \omega, t) \cdot \Phi_m(j, \omega, t)$, where

[6] Zero equilibrium R&D by incumbents is a well-known result claimed by the traditional quality-ladders models (e.g., Aghion and Howitt, 1992). However, as shown by Cozzi (2007a), the assumption of R&D firms (potential entrants and the incumbent) operating under perfect competition and constant returns at the firm level, taken rigorously, "yields an indeterminate investment for the incumbent, thereby predicting that incumbents should invest randomly", which is consistent with the latter doing any amount of R&D, from zero to a very large number. Our assumption of zero equilibrium R&D by incumbents is only for the sake of simplicity in what regards the microstructure of our model.

the function Φ is the same for every firm in ω and captures the effect of the current technological-knowledge position j. Now, let us define

$$\begin{aligned} \Phi_L(j,\omega,t) &\equiv \tfrac{1}{\zeta}\,(lL)^{-\epsilon}\,\lambda^{-(j_L(\omega,t)+1)\left(\frac{1-\alpha}{\alpha}\right)} \\ \Phi_H(j,\omega,t) &\equiv \tfrac{1}{\zeta}\,(hH)^{-\epsilon}\,\lambda^{-(j_H(\omega,t)+1)\left(\frac{1-\alpha}{\alpha}\right)} \end{aligned} \tag{20}$$

where $\zeta > 0$ is a constant that stands for the (flow) fixed vertical-R&D cost (for simplicity, we assume $\zeta_L \equiv \zeta_H \equiv \zeta$) and $\epsilon \geq 0$ is a parameter that allows for a varying degree of scale-effects removal associated to the size of the labour force measured in efficiency units. We assume that an increase in market scale dilutes the effect of R&D outlays on innovation probability, in line with, e.g., Barro and Sala-i-Martin (2004), capturing the idea that the difficulty of introducing new intermediate goods and replacing old ones is proportional to the size of the market, measured by labour employed. The reasons this may happen are coordenation, organisational and transportation costs (e.g., Dinopoulos and Thompson, 1999) or rental protection actions by incumbents (e.g., Sener, 2008), also expected to be proportional to market size. One can conceive ϵ within any range in the interval $[0,\infty)$, depending on whether the benefits of scale, connected to the size of profits that accrue to the incumbent each t (see (16)), are either not affected ($\epsilon = 0$) or are partial ($0 < \epsilon < 1$), totally ($\epsilon = 1$) or over ($\epsilon > 1$) couterbalanced by the referred costs and rental protection actions associated to market size. Also, implicit in (20) is the assumption of *dynamic* decreasing returns to scale to vertical R&D (i.e., decreasing returns to *cumulated* R&D).[7,8]By aggregating across firms in ω, we get $R_{vm}(j,\omega,t) = \sum_i R^i_{vm}(j,\omega,t)$ and $I_m(j,\omega,t) = \sum_i I^i_m(j,\omega,t)$, such that (shown for $m = L$)

$$I_L(j,\omega,t) = R_{vL}(j,\omega,t)\cdot\frac{1}{\zeta}\cdot(lL)^{-\epsilon}\cdot\lambda^{-(j_L(\omega,t)+1)\left(\frac{1-\alpha}{\alpha}\right)} \tag{21}$$

From (21), we can aggregate across ω to get the total resources devoted to vertical R&D, $R_{vm}(t)$, for a given $N_m(t)$, (for $m = L$)

$$R_{vL}(t) = \int_0^{N_L(t)} R_{vL}(j,\omega,t)d\omega = \int_0^{N_L(t)} \zeta\cdot(lL)^{\epsilon}\cdot\lambda^{(j_L(\omega,t)+1)\left(\frac{1-\alpha}{\alpha}\right)}\cdot I_L(j,\omega,t)d\omega \tag{22}$$

[7]The specific way Φ depends on j implies that the increasing difficulty of creating new product generations over time exactly offsets the increased rewards from marketing higher quality products; see (20) and (16). This allows for constant probability over time and across industries in balanced-growth path, i.e., a symmetric equilibrium (on *asymmetric* equilibrium in quality-ladders models and its growth consequences, see Cozzi, 2007b).

[8]Sener (2008) contrasts the effects of rental protection actions with the expanding variety and the dynamic decreasing returns to R&D as scale-removal mechanisms within a quality-ladders model with knowledge-driven R&D specification. Observe, however, that the dynamic decreasing returns to R&D, as first introduced by Segerstrom (1998), and represented in our model by the term $\lambda^{-(j_L(\omega,t)+1)\left(\frac{1-\alpha}{\alpha}\right)}$ in (20), are neither necessary nor sufficient for the purpose of scale removal in a model of lab-equipement specification as our own (though it plays a crucial role in guaranteeing a Poisson rate constant over ω and hence the existence of a symmetric equilibrium; see fn. 7, above). The same applies to the expanding variety mechanism, as it is clear if we let $\epsilon = 0$ in our results below.

Taking $I_m(j,\omega,t)$ as given, it defines the probability of the incumbent losing his monopoly position. Thus, the present value of an incumbent's profits is a random variable because the terminal date to the monopoly of firm i arrives with probability $I_m(j,\omega,t)$ per (infinitesimal) increment of time. Let $V_m(j,\omega,t)$ denote the *expected* discounted value of profits earned by a monopolist when the highest quality in m-complementary ω is j.[9] V can be interpreted as the market value of the patent or the value of the monopolist firm owned by households. The expected discounted value of profits can be written as $V_m(j,\omega,t) = \int_t^\infty \pi_m(j,\omega,t) e^{-\int_t^s (r(v)+I_m(j,\omega,v))dv} ds$, where r is the equilibrium market real interest rate and $\pi_m(j,\omega,t)$ is given by (16). The equation above reflects the fact that, if a profit flow can stop when a Poisson event with arrival rate I occurs, then we can calculate the *expected* present value of the stream of profit as if it never stops, but adding I to the discount rate. Thus, we can interpret $r+I$ as an *effective* discount rate. Since $\pi_m P_m^{-\frac{1}{\alpha}}$ is constant in-between innovations, we can further write

$$V_m(j,\omega,t) = \bar{\pi}_m \lambda^{j_m(\omega,t)\left(\frac{1-\alpha}{\alpha}\right)} \int_t^\infty P_m(t)^{\frac{1}{\alpha}} e^{-\int_t^s (r(v)+I_m(j,\omega,v))dv} ds \tag{23}$$

Now, consider the average m-complementary intermediate-good sector, $\bar{\omega}$, for a given $N_m(t)$.[10] Average resources devoted to vertical R&D, $R_{vm}(j,\bar{\omega},t) = \frac{R_{vm}(t)}{N_m(t)}$, can be put into (21) to yield an expression for the probability of vertical innovation for $\bar{\omega}$, $I_m(j,\bar{\omega},t)$. With free-entry into the vertical R&D business, we have the free-entry condition

$$I_m(j,\bar{\omega},t) \cdot V_m(j+1,\bar{\omega},t) = R_{vm}(j,\bar{\omega},t) \tag{24}$$

By substituting (23) into (24), we get

$$I_m(j,\bar{\omega},t) \cdot \bar{\pi}_m \cdot \lambda^{j_m(\omega,t)\left(\frac{1-\alpha}{\alpha}\right)} \int_t^\infty P_m(t)^{\frac{1}{\alpha}} \cdot e^{-\int_t^s (r(v)+I_m(j,\omega,v))dv} ds = R_{vm}(j,\bar{\omega},t) \tag{25}$$

whilst time-differentiating (25), bearing in mind Leibniz's rule, yields (henceforth, the dot denotes time derivative)

$$r(t) + I_m(j+1,\bar{\omega},t) = \frac{\pi_m(j+1,\bar{\omega},t) \cdot I_m(j,\bar{\omega},t)}{R_{vm}(j,\bar{\omega},t)} -$$

$$-\left(\frac{\dot{\pi}_m(j+1,\bar{\omega},t)}{\pi_m(j+1,\bar{\omega},t)} - \frac{1}{\alpha}\frac{\dot{P}_m(t)}{P_m(t)}\right) - \frac{\dot{I}_m(j,\bar{\omega},t)}{I_m(j,\bar{\omega},t)} + \frac{\dot{R}_{vm}(j,\bar{\omega},t)}{R_{vm}(j,\bar{\omega},t)} \tag{26}$$

This can be interpreted as an arbitrage condition, which *equates the effective rate of return on capital* (i.e., the market rate of return augmented by the Poisson arrival rate)

[9] We assume that entrants are risk-neutral and, thus, only care about the expected value of the firm.

[10] The usual procedure in the quality-ladders literature is to consider the average intermediate-good sector in order to avoid any jumpiness in quality levels that would occur if the behaviour of an individual sector were contemplated.

to the rate of return on vertical R&D, where the latter equals the profit rate earned by setting up now a new firm with an existing intermediate good of improved quality minus the increase in the profit rate due to the next innovation in that intermediate good (which is accrued to the next innovator).[11]As a result of (16) and (21) applied to $\bar{\omega}$, we have, after time differentiation,

$$\frac{\dot{\pi}_m(j+1,\bar{\omega},t)}{\pi_m(j+1,\bar{\omega},t)} = \frac{\dot{\pi}_m(j,\bar{\omega},t)}{\pi_m(j,\bar{\omega},t)} = I_m(j,\bar{\omega},t)\cdot\left[\dot{j}_m(\bar{\omega},t)\cdot\left(\frac{1-\alpha}{\alpha}\right)\cdot ln\lambda\right] + \frac{1}{\alpha}\frac{\dot{P}_m(t)}{P_m(t)} \quad (27)$$

and

$$\frac{\dot{R}_{vm}(j,\bar{\omega},t)}{R_{vm}(j,\bar{\omega},t)} = \frac{\dot{I}_m(j,\bar{\omega},t)}{I_m(j,\bar{\omega},t)} + I_m(j,\bar{\omega},t)\cdot\left[\dot{j}_m(\bar{\omega},t)\cdot\left(\frac{1-\alpha}{\alpha}\right)\cdot ln\lambda\right] \quad (28)$$

Hence, given (28) and (27), we can rewrite (26) as

$$r(t) + I_m(j+1,\bar{\omega},t) = \frac{\pi_m(j+1,\bar{\omega},t)\cdot I_m(j,\bar{\omega},t)}{R_{vm}(j,\bar{\omega},t)} \quad (29)$$

Substituting (16) and (21) in the right-hand side of (29), and following the same steps for $m = H$, yields

$$\begin{array}{l} r(t) = \frac{\bar{\pi}_L\cdot P_L(t)^{\frac{1}{\alpha}}}{\zeta\cdot(lL)^{\epsilon}} - I_L(j+1,\bar{\omega},t) \Leftrightarrow r(t) = \frac{\bar{\pi}\cdot(lL)^{1-\epsilon}\cdot P_L(t)^{\frac{1}{\alpha}}}{\zeta} - I_L(t) \\ r(t) = \frac{\bar{\pi}_H\cdot P_H(t)^{\frac{1}{\alpha}}}{\zeta\cdot(hH)^{\epsilon}} - I_H(j+1,\bar{\omega},t) \Leftrightarrow r(t) = \frac{\bar{\pi}\cdot(hH)^{1-\epsilon}\cdot P_H(t)^{\frac{1}{\alpha}}}{\zeta} - I_H(t) \end{array} \quad (30)$$

where $\bar{\pi} \equiv \frac{\bar{\pi}_L}{lL} = \frac{\bar{\pi}_H}{hH}$. According to (30), the relationship between r and I_m is independent of t, ω, and j, implying $I_m(t) \equiv I_m(j+1,\bar{\omega},t)$. Thus, if I_m is constant over time, then r is also constant.

Horizontal R&D free-entry and dynamic arbitrage conditions

Variety expansion results from R&D aimed at creating a new intermediate-good line, corresponding to a new firm, at a cost of η units of final output. In particular, we view the creation of new product lines as a product development activity without positive spillovers and allow for entry as well as exit of product lines from the market - that is, we do not assume irreversibility of investment in product development.

After a new product is launched, an initial quality level is observed, drawn at random from the distribution of quality indexes matching the existing product lines. Let $q_m(j,\omega,t) \equiv \lambda^{j_m(\omega,t)\left(\frac{1-\alpha}{\alpha}\right)}$ be an alternative measure of product quality in the technology group $m = L, H$.Then, from (5), we have

$$Q_m(t) = \int_0^{N_m(t)} q_m(j,\omega,t)d\omega = q_m(j,\bar{\omega},t)\cdot N_m(t) \quad (31)$$

[11]In (26), $\frac{\dot{\pi}_m}{\pi_m}$ (as well as $\frac{\dot{R}_{vm}}{R_{vm}}$ and $\frac{\dot{I}_m}{I_m}$) must be interpreted in *expected* terms, since it reflects the stochastic process of innovation arrival.

where $q_m(j,\bar{\omega},t) \equiv E_\omega(q_m)$ is the average of q over industries.[12] Assume perfect competition among R&D firms and constant returns to scale at the firm level, such that $\dot{N}^e_m(t) = \frac{1}{\eta} \cdot R^e_{hm}(t)$, where $\dot{N}^e_m(t)$ is the contribution to the instantaneous flow of new m-complementary intermediate goods by innovator firm e and $R^e_{hm}(t)$ is the flow of resources devoted to horizontal R&D by e at t (measured in units of final-good output Y). The cost η_m is the same for every firm doing horizontal R&D in the m-complementary sector. Next, aggregate across firms to get $R_{hm}(t) = \sum_e R^e_{hm}(t)$ and $\dot{N}_m(t) = \sum_e \dot{N}^e_m(t)$, which implies that the total flow of resources devoted to horizontal R&D is

$$R_{hm}(t) = \eta_m \cdot \dot{N}_m(t) \tag{32}$$

Since entry also generates value $V_m(q_m(j,\bar{\omega},t)) \equiv V_m(j,\bar{\omega},t)$, a free-entry equilibrium requires that new product lines are created (or destroyed) at a rate $\dot{N}_m$ necessary to ensure that

$$\dot{N}_m(t) \cdot V_m(j,\bar{\omega},t) = R_{hm}(t) \Leftrightarrow V_m(j,\bar{\omega},t) = \eta_m \tag{33}$$

Henceforth, we explore the case where η_m is time-varying. In particular, let

$$\eta_m \equiv \eta(N_m, R_{hm}) = \varphi_1(N_m) \cdot \varphi_2(R_{hm}) \tag{34}$$

where $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ are positive, invertible functions. This specification of the entry cost function merges the one suggested by Romer (1990) and Barro and Sala-i-Martin (2004, ch. 6) (through the first term in the right-hand side of (34)), by which the entry cost increases with the number of differentiated goods in the market, thus implying decreasing returns to *cumulated* horizontal R&D, with the one by Howitt (1999) (the second term), where the aggregate function of horizontal innovation exhibits decreasing marginal returns in the total *flow* of horizontal R&D.[13] Let $\varphi_1(N_m) = \psi N_m(t)^{\nu_1}$ and $\varphi_2(R_{hm}) = R_{hm}(t)^{\nu_2}$, where the parameter $\psi > 0$ stands for a (flow) fixed horizontal-R&D cost, $\nu_1 > 0$ measures the negative spillover effect related to the accumulation of intermediate-good varieties, whilst $0 < \nu_2 < 1$ measures the degree of the increasing returns of the *marginal* horizontal innovation function, $\varphi_2(R_{hm})^{-1}$ (for simplicity, we assume the parameters take the same values in the two technology groups). By substituting (32) in (34), we arrive at

[12] Hence, from (15) and (16), we have $X_m(q_m(j,\bar{\omega},t)) \equiv X_m(j,\bar{\omega},t)$ and $\pi_m(q_m(j,\bar{\omega},t)) \equiv \pi_m(j,\bar{\omega},t)$.

[13] The dependence of η on R_h implies (*static*) decreasing returns to scale to horizontal R&D at the aggregate level, but which we assume to be entirely external to the firm, and hence compatible with the previous assumption of constant returns to scale at the firm level. This is a departure from Howitt (1999) (see also Segerstrom, 2000) - who assume decreasing returns to scale to R&D at the firm level - but is in line with, e.g., Arnold (1998). On the other hand, observe that the *dynamic* decreasing returns to scale to horizontal R&D due to the dependence of η on N parallel the *dynamic* decreasing returns to vertical R&D due to the dependence of Φ on λ^j (see (20); also note, from (24), that free entry in vertical R&D implies $V = \frac{1}{\Phi}$) or, more precisely, of $\int_0^{N(t)} \Phi(j,\omega,t)d\omega$ on $q(j,\bar{\omega},t)N(t) = Q(t)$, as shown below (see (38)).

$$\eta(\cdot) = \psi \cdot N_m(t)^{\nu_1} \cdot \Big(\eta(\cdot) \cdot \dot{N}_m(t)\Big)^{\nu_2} \Leftrightarrow \eta(\cdot) = \psi^{\frac{1}{1-\nu_2}} \cdot N_m(t)^{\frac{\nu_1}{1-\nu_2}} \cdot \dot{N}_m(t)^{\frac{\nu_2}{1-\nu_2}} = \phi \cdot N_m(t)^{\sigma} \cdot \dot{N}_m(t)^{\gamma} \tag{35}$$

where $\phi \equiv \psi^{\frac{1}{1-\nu_2}} > 0$, $\sigma \equiv \frac{\nu_1}{1-\nu_2} > 0$ and $\gamma \equiv \frac{\nu_2}{1-\nu_2} > 0$. Equation (35) shows the link between our specification of the horizontal entry-cost function with respect to R_h and that used by Datta and Dixon (2002) and Brito and Dixon (2008), where the entry cost increases with the number of goods entering the market at a given instant, $\dot{N}$.[14]This mechanism, which introduces dynamic second-order effects in entry, is also similar to the one that characterises the changes of the physical-capital stock in the literature of firm investment with convex adjustment costs, where the cost of installing (dismantling) capital increases with the amount of investment (disinvestment) at a given instant (e.g., Eisner and Strotz, 1963).

By substituting (23) into (33), where η_m is a time-varying function, we have

$$\bar{\pi}_m \cdot \lambda^{j_m(\omega,t)\left(\frac{1-\alpha}{\alpha}\right)} \cdot \int_t^{\infty} P_m(t)^{\frac{1}{\alpha}} \cdot e^{-\int_t^s (r(v)+I_m(j,\omega,v))dv} ds = \eta_m \tag{36}$$

If we time-differentiate (36), assuming η_m is differentiable with respect to time, we get

$$r(t) + I_m(j,\bar{\omega},t) = \frac{\pi_m(j,\bar{\omega},t)}{\eta_m} - \left(\frac{\dot{\pi}_m(j,\bar{\omega},t)}{\pi_m(j,\bar{\omega},t)} - \frac{1}{\alpha}\frac{\dot{P}_m(t)}{P_m(t)}\right) + \frac{\dot{\eta}_m}{\eta_m} \tag{37}$$

This is another arbitrage equation, according to which *the effective rate of return on capital equals the rate of return on horizontal R&D, where the latter equals the profit rate earned by setting up now a new firm with a new product line minus the increase in the profit rate due to the next vertical innovation in that intermediate good* (which is accrued to the next innovator).[15]

Consistency arbitrage conditions

Finally, a consistency condition between vertical and horizontal arbitrage conditions is needed. First, we find an expression for $R_{vm}(j-1,\bar{\omega},t)$, by applying (22) to $j-1$ (the same for $m=H$) and combining it with (31), for a given $N_m(t)$, (shown for $m=L$)

$$R_{vL}(j-1,\bar{\omega},t) = \frac{\int_0^{N_L(t)} R_{vL}(j-1,\omega,t)d\omega}{N_L(t)} = \frac{I_L(t)\cdot\zeta\cdot(lL)^{\epsilon}\cdot Q_L(t)}{N_L(t)} = I_L(t)\cdot\zeta\cdot(lL)^{\epsilon}\cdot q_L(j,\bar{\omega},t) \tag{38}$$

[14]However, observe that, given our assumption of constant returns to scale at the firm level, $V = \eta = \frac{R_h}{\dot{N}} = \frac{dR_h}{d\dot{N}}$ (see (33)). At this point, we departure from the model of entry in Brito and Dixon (2008), as these authors implicitly assume an entry technology with decreasing returns, with the result $V = \eta = \frac{dR_h}{d\dot{N}} > \frac{R_h}{\dot{N}}$. This is equivalent to the model of entry in Howitt (1999) and Segerstrom (2000) (see fn. 13, above). Note that the price of entry (V) equals the marginal cost of entry (η) in all cases considered above; nevertheless, our assumption of constant returns eschews positive profits from entering, since $V = \frac{R_h}{\dot{N}}$.

[15]Remember that $\frac{\dot{\pi}_m}{\pi_m}$, in (37), must be interpreted in *expected* terms.

where we used $I_L(t) \equiv I_L(j-1, \bar{\omega}, t)$. Then, from the vertical free-entry condition, (24), solved in order to V_m, we get $V_L(j+1, \bar{\omega}, t) = \frac{R_{vL}(j,\bar{\omega},t)}{I_L(j,\bar{\omega},t)} \Rightarrow V_L(j, \bar{\omega}, t) = \frac{R_{vL}(j-1,\bar{\omega},t)}{I_L(j-1,\bar{\omega},t)}$. Together with (38), we have (for $m = L$)

$$V_L(j, \bar{\omega}, t) = \zeta \cdot (lL)^{\epsilon} \cdot q_L(j, \bar{\omega}, t) \tag{39}$$

At last, equating (39) and the horizontal free-entry condition, (33), and following the same steps for $m = H$, yields the consistency conditions

$$\begin{aligned} q_L(j, \bar{\omega}, t) &= \frac{Q_L(t)}{N_L(t)} = \frac{\eta_L}{\zeta \cdot (lL)^{\epsilon}} \\ q_H(j, \bar{\omega}, t) &= \frac{Q_H(t)}{N_H(t)} = \frac{\eta_H}{\zeta \cdot (hH)^{\epsilon}} \end{aligned} \tag{40}$$

These conditions imply that, for each $m = L, H$, both the real rate of return to vertical R&D and to horizontal R&D equal r for every t, meaning that the (competitive) capital market is always willing to finance both activities.[16]

2.3. The consumer sector

The economy consists of $L+H$ identical dynastic families who consume and collect income (dividends) from investments in financial assets (equity) and labour income. They choose the path of final-good aggregate consumption $\{C(t), t \geq 0\}$ to maximise discounted lifetime utility

$$U = \int_0^{\infty} \left(\frac{C(t)^{1-\theta} - 1}{1-\theta} \right) e^{-\rho t} dt \tag{41}$$

where $\rho > 0$ is the subjective discount rate and $\theta > 0$ is the constant elasticity of marginal utility with respect to consumption. We assume consumers have perfect foresight concerning the aggregate rate of technological change over time and choose their expenditure paths accordingly to maximise their discounted utilities, dispensing with the time expectations operator, $E(.)$, in (41).

Intertemporal utility is maximised subject to the flow budget constraint

$$\dot{a}(t) = r(t) \cdot a(t) + w_L(t) \cdot L + w_H(t) \cdot H - C(t) \tag{42}$$

where a stands for households' financial assets (equity) holdings, measured in terms of final-good output Y. Households take the real rate of return on financial assets, r (that is, dividend payments in units of asset price corrected by the Poisson death rate, $r = \frac{\pi_m}{V_m} - I_m$)[17] and the real labour wage, w_m, as given. The initial level of wealth $a(0)$ is also given, whereas the condition $lim_{t \to \infty} e^{-\int_0^t r(s) ds} a(t) \geq 0$ is imposed in order to prevent Ponzi schemes.

[16] Observe also that by time-differentiating (40), we get $\frac{\dot{\eta}_m}{\eta_m} = \frac{\dot{\pi}_m}{\pi_m} - \frac{1}{\alpha} \frac{\dot{P}_m}{P_m}$. Substituting the latter in (37) yields (30).

[17] This equation can be read as an arbitrage condition for investors, which requires that the real interest rate equals the dividend rate, $\frac{\pi_m}{V_m}$, plus the rate of capital gain, $-I_m$. This condition can be derived, e.g., by solving (24) in order to V_m and substituting the result in (29).

The optimal path of consumption satisfies the well-known differential Euler equation

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}\left(r(t) - \rho\right) \tag{43}$$

as well as the transversality condition

$$\lim_{t\to\infty} e^{-\rho t} C(t)^{-\theta} a(t) = 0 \tag{44}$$

3. General equilibrium

In this section, we construct the general equilibrium and characterise the interior steady state of our model.

3.1. The aggregate resource constraint

The balance sheet of households equates the value of equity holdings to the market value of firms, that is

$$a(t) \equiv a_L(t) + a_H(t) = V_H(j,\bar{\omega},t)\cdot N_H(t) + V_L(j,\bar{\omega},t)\cdot N_L(t) = \eta_H \cdot N_H(t) + \eta_L \cdot N_L(t) \tag{45}$$

Hence, we can characterise the change in the value of equity as

$$\dot{a}(t) = \eta_H \cdot \dot{N}_H(t) + \dot{\eta}_H \cdot N_H(t) + \eta_L \cdot \dot{N}_L(t) + \dot{\eta}_L \cdot N_L(t) \tag{46}$$

Substituting (45), (42) and (37) (the latter solved in order to $\dot{\eta}_m$) in (46), yields, after some algebric manipulation,

$$Y(t) = X(t) + C(t) + R_h(t) + R_v(t) \tag{47}$$

where $R_h = R_{hH} + R_{hL}$ and $R_v = R_{vH} + R_{vL}$.[18] Equation (47) tells us that total final-good output, Y, is allocated among total consumption, C, total production of intermediate goods, X, total vertical R&D expenditures, R_v, and total horizontal R&D expenditures, R_h, thus being a *product market equilibrium* equation.

Next, note that, by using $I_L(t) \equiv I_L(j,\omega,t)$ in (22), we get

$$R_{vL}(t) = \int_0^{N_L(t)} \Phi_L(j,\omega,t)^{-1} \cdot I_L(j,\omega,t) d\omega = I_L(t)\cdot \zeta \cdot (lL)^{\epsilon} \cdot \lambda^{\frac{1-\alpha}{\alpha}} \cdot Q_L(t) \tag{48}$$

An equivalent expression obtains for $m = H$.[19] We use the latter, together with (17), (18) and (32), to re-write (47) as

[18] The algebraic expressions for R_{hm} and R_{vm} implicit in (47) are $R_{hm} = \dot{\eta}_m \cdot N_m$ and $R_{vm} = I_m \cdot a_m + \left(\frac{\dot{\pi}_m}{\pi_m} - \frac{1}{\alpha}\frac{\dot{P}_m}{P_m}\right) a_m$, where $a_m = V_m \cdot N_m$. See Appendix B for a detailed derivation.

[19] It can be shown that the consistency between the expression for R_v implicit in (47) (see fn. 18) and (48) is guaranteed by (40).

$$\chi_Y \left[P_H(t)^{\frac{1}{\alpha}} \cdot h \cdot H \cdot Q_H(t) + P_L(t)^{\frac{1}{\alpha}} \cdot l \cdot L \cdot Q_L(t) \right] =$$

$$= \chi_X \left[P_H(t)^{\frac{1}{\alpha}} \cdot h \cdot H \cdot Q_H(t) + P_L(t)^{\frac{1}{\alpha}} \cdot l \cdot L \cdot Q_L(t) \right] + C(t) + \eta_L \cdot \dot{N}_L(t) + \eta_H \cdot \dot{N}_H(t) +$$

$$+ I_L(t) \cdot \zeta \cdot (lL)^{\epsilon} \cdot \lambda^{\frac{1-\alpha}{\alpha}} \cdot Q_L(t) + I_H(t) \cdot \zeta \cdot (hH)^{\epsilon} \cdot \lambda^{\frac{1-\alpha}{\alpha}} \cdot Q_H(t) \tag{49}$$

where $\chi_Y \equiv A^{\frac{1}{\alpha}} (1-\alpha)^{\frac{2(1-\alpha)}{\alpha}}$ and $\chi_X \equiv A^{\frac{1}{\alpha}} (1-\alpha)^{\frac{2}{\alpha}}$.

3.2. The dynamic system

The general equilibrium is defined by the system of nine equations: the Euler equation for consumption (43); the households' transversality condition (44); the horizontal arbitrage conditions (37); the vertical arbitrage conditions (30); the arbitrage consistency conditions (40); the product market equilibrium equation (49), plus the necessary initial conditions.

We wish to obtain the dynamic system for Q_m, N_m and C. Firstly, we solve (40) in order to $\dot{N}_m$, to obtain an ordinary differential equation (ODE) in N_m. Secondly, by assuming that the number of sectors, N_m, is large enough to treat Q_m as time-differentiable and the time interval dt is small enough to have $\dot{Q}_m$ non-stochastic, we time differentiate (31) and simplifiy it with the previously obtained ODE in $N_m$to get an ODE in Q_m.Together with the Euler equation for consumption, (43), the dynamic system reads

$$\dot{N}_m(t) = x_m(Q_m, N_m) \cdot N_m(t) \tag{50a}$$

$$\dot{Q}_m(t) = [\Xi \cdot I_m(t) + x_m(Q_m, N_m)] \cdot Q_m(t) \tag{50b}$$

$$\dot{C}(t) = \frac{1}{\theta} \cdot (r(t) - \rho) \cdot C(t) \tag{50c}$$

where $\Xi \equiv \left(\lambda^{\frac{1-\alpha}{\alpha}} - 1 \right)$ and

$$x_L(Q_L, N_L) \equiv \left(\frac{\zeta}{\phi} \cdot (lL)^{\epsilon} \right)^{\frac{1}{\gamma}} \cdot Q_L(t)^{\frac{1}{\gamma}} \cdot N_L(t)^{-\left(\frac{\sigma+\gamma+1}{\gamma}\right)} \tag{51}$$

$$x_H(Q_H, N_H) \equiv \left(\frac{\zeta}{\phi} \cdot (hH)^{\epsilon} \right)^{\frac{1}{\gamma}} \cdot Q_H(t)^{\frac{1}{\gamma}} \cdot N_H(t)^{-\left(\frac{\sigma+\gamma+1}{\gamma}\right)} \tag{52}$$

Equations (50a)-(50c) define a system of five non-linear ODE's, where r and I_L (or I_H) are connected through (30), and $I_L$and I_H are connected through

$$I_H(t) - I_L(t) = \frac{\bar{\pi}}{\zeta} \left[(hH)^{1-\epsilon} \cdot P_H(t)^{\frac{1}{\alpha}} - (lL)^{1-\epsilon} \cdot P_L(t)^{\frac{1}{\alpha}} \right] \tag{53}$$

The latter is an inter-technology arbitrage condition obtained by equating the two equations in (30), where we aknowledge the fact that the country's interest rate is always unique.

In turn, solve (49), e.g., in order to I_L, take (40) to eliminate η_m and simplify across with (50a), to get

$$I_L(t) \equiv I_L(Q_L, Q_H, N_L, N_H, C, I_H) =$$

$$= \frac{1}{\zeta \cdot \lambda^{\frac{1-\alpha}{\alpha}} \cdot (lL)^{\epsilon}} (\chi_Y - \chi_X) \left[P_H(t)^{\frac{1}{\alpha}} \cdot h \cdot H \cdot \frac{Q_H(t)}{Q_L(t)} + P_L(t)^{\frac{1}{\alpha}} \cdot l \cdot L \right] - \left(\frac{hH}{lL} \right)^{\epsilon} \cdot \frac{Q_H(t)}{Q_L(t)} \cdot I_H(t) -$$

$$- \frac{1}{\lambda^{\frac{1-\alpha}{\alpha}}} \cdot \left(\frac{hH}{lL} \right)^{\epsilon} \cdot \frac{Q_H(t)}{Q_L(t)} \cdot x_H(Q_H, N_H) - \frac{1}{\lambda^{\frac{1-\alpha}{\alpha}}} \cdot x_L(Q_L, N_L) - \frac{1}{\zeta \cdot \lambda^{\frac{1-\alpha}{\alpha}} \cdot (lL)^{\epsilon}} \cdot \frac{C(t)}{Q_L(t)} \quad (54)$$

If we further use (53) to eliminate I_H from (50b)-(50c) and (54), to get $I_L(t) \equiv I_L(Q_L, Q_H, N_L, N_H, C)$, and since P_L and P_H are (non-linear) functions of $\frac{Q_H}{Q_L}$ alone (see (14)), we are able to define the system of five ODE's $\dot{N}_L = F_{N_L}(Q_L, N_L)$, $\dot{N}_H = F_{N_H}(Q_H, N_H)$, $\dot{Q}_L = F_{Q_L}(Q_L, Q_H, N_L, N_H, C)$, $\dot{Q}_H = F_{Q_H}(Q_L, Q_H, N_L, N_H, C)$ and $\dot{C} = F_C(Q_L, Q_H, N_L, N_H, C)$.

3.3. The steady state

Now, we derive and characterise the interior steady-state equilibrium. First, it is convenient to find a transformation of the system (50a)-(50c) such that we can work with an equivalent system whose equilibria are fixed points. The stability and unicity of the interior steady-state equilibrium are shown within this framework.

Proposition 1. Let $g_y \equiv \dot{y}/y$, the growth rate of a variable y along the balanced growth path. In this model, steady-state equilibria have the following characteristics: (i) $g_C = g_{Q_L} = g_{Q_H} = g$; (ii) $g_{I_L} = g_{I_H} = 0$; (iii) $g_{P_L} = g_{P_H} = 0$; (iv) $\frac{g_{Q_m}}{g_{N_m}} = (\sigma + \gamma + 1)$, $x_m \neq 0$; and (v) $g_{N_L} = g_{N_H}$.

Proof: See Appendix C.

Having the above in mind, we transform the system (50a)-(50c) into a system of rescaled variables. Recall (51)-(52) and let

$$z_L(t) = \frac{C(t)}{Q_L(t)} \quad (55)$$

$$Q^*(t) = \frac{Q_H(t)}{Q_L(t)} \quad (56)$$

with the property that, in the steady state, $\dot{x} = \dot{z} = \dot{Q}^* = 0$. After time-differentiating (51),(52), (55) and (56), and substituting with (50a), (50b), (50c), (30) (solved in order to r) and (53) (solved in order to I_H) where necessary, we get the system

$$\dot{x}_L(t) = \left[\frac{\Xi}{\gamma} \cdot I_L(t) - \left(\frac{\sigma+\gamma}{\gamma}\right) \cdot x_L(t)\right] \cdot x_L(t) \tag{57a}$$

$$\dot{z}_L(t) = \left\{\frac{1}{\theta} \cdot \left[\frac{\bar{\pi}}{\zeta} \cdot (lL)^{1-\epsilon} \cdot P_L(t)^{\frac{1}{\alpha}} - \rho\right] - \left(\frac{1}{\theta} + \Xi\right) \cdot I_L(t) - x_L(t)\right\} \cdot z_L(t) \tag{57b}$$

$$\dot{x}_H(t) = \left\{\frac{\Xi}{\gamma} \cdot I_L(t) - \left(\frac{\sigma+\gamma}{\gamma}\right) \cdot x_H(t) + \frac{\Xi}{\gamma} \cdot \frac{\bar{\pi}}{\zeta}\left[(hH)^{1-\epsilon} \cdot P_H(t)^{\frac{1}{\alpha}} - (lL)^{1-\epsilon} \cdot P_L(t)^{\frac{1}{\alpha}}\right]\right\} \cdot x_H(t) \tag{57c}$$

$$\dot{Q}^*(t) = \left\{\Xi \cdot \frac{\bar{\pi}}{\zeta} \cdot \left[(hH)^{1-\epsilon} \cdot P_H(t)^{\frac{1}{\alpha}} - (lL)^{1-\epsilon} \cdot P_L(t)^{\frac{1}{\alpha}}\right] + x_H(t) - x_L(t)\right\} \cdot Q^*(t) \tag{57d}$$

where $I_L(t) \equiv I_L(x_L, x_H, z_L, Q^*) \equiv I_L(Q_L, Q_H, N_L, N_H, C)$, as we can see by substituting (53) (solved in order to I_H), (55) and (56) in (54). Since P_L and P_H are (non-linear) functions of Q^* alone (see (14) with (56)), we have a system of rescaled variables equivalent to (50a)-(50c) which comprises four ODE's, $\dot{x}_L = F_{x_L}(x_L, x_H, z_L, Q^*) \cdot x_L$, $\dot{z}_L = F_{z_L}(x_L, x_H, z_L, Q^*) \cdot z_L$, $\dot{x}_H = F_{x_H}(x_L, x_H, z_L, Q^*) \cdot x_H$ and $\dot{Q^*}(t) = F_{Q^*}(x_L, x_H, Q^*) \cdot Q^*$. This system connects the dynamics of Q_m with N_m $(m = H, L)$, C with Q_L and Q_L with Q_H, displaying one jump variable (z_L) and three pre-determined variables. Equations (57a)-(57d), plus the transversality condition and the initial conditions $x_L(0)$, $x_H(0)$ and $Q^*(0)$ describe the transitional dynamics and the steady state of the model, by jointly determining the variables $(x_L(t), z_L(t), x_H(t), Q^*(t))$. From these we can determine the original variables $N_L(t)$, $N_H(t)$, $C(t)$ and $Q_L(t)$ (alternatively, $Q_H(t)$), for a given $Q_H(t)$ ($Q_L(t)$). That is, the system is undetermined in $Q_H(t)$ ($Q_L(t)$).

As usual, the fixed points of the system are found by equating $\dot{x}_L = 0$, $\dot{z}_L = 0$, $\dot{x}_H = 0$ and $\dot{Q}^* = 0$. However, given the special structure of our model in steady state, we can solve first for the latter equation and use the result to solve jointly $\dot{z}_L = 0$ and $\dot{x}_L = 0$. The solution for $\dot{x}_H = 0$ follows directly from the latter.

We are interested in the interior steady state, i.e., $\tilde{x}_L \neq 0 \ \wedge \ \tilde{z}_L \neq 0 \ \wedge \ \tilde{x}_H \neq 0 \ \wedge \ (\tilde{Q^*}) \neq 0$, where ˜ indicates steady-state value. Given Proposition 1 and (50a), we know that $\tilde{x}_L = \tilde{x}_H$. Together with (57d), we find that $\dot{Q}^* = 0$ implies

$$(hH)^{1-\epsilon} P_H^{\frac{1}{\alpha}} - (lL)^{1-\epsilon} P_L^{\frac{1}{\alpha}} = 0 \Leftrightarrow (\tilde{P^*}) \equiv \left(\tilde{\frac{P_H}{P_L}}\right) = \left(\frac{lL}{hH}\right)^{\alpha(1-\epsilon)} \tag{58}$$

Note that this result also guarantees $\tilde{I}_L = \tilde{I}_H$ (see (53)). Next, substitute (58) in (12) and solve in order to $\frac{Q_H}{Q_L}$ to get

$$(\tilde{Q^*}) \equiv \left(\tilde{\frac{Q_H}{Q_L}}\right) = \left(\frac{hH}{lL}\right)^{1-2\epsilon} \tag{59}$$

From here, together with (13), (14) and (19), we find

$$\tilde{P}_L = e^{-\alpha}\left[1+\left(\frac{hH}{lL}\right)^{1-\epsilon}\right]^{\alpha};\quad \tilde{P}_H = e^{-\alpha}\left[1+\left(\frac{lL}{hH}\right)^{1-\epsilon}\right]^{\alpha} \tag{60}$$

$$\tilde{\tilde{n}} = \left[1+\left(\frac{hH}{lL}\right)^{1-\epsilon}\right]^{-1} \tag{61}$$

$$\widetilde{\left(\frac{w_H}{w_L}\right)} = \left(\frac{h}{l}\right)^{1-\epsilon}\left(\frac{H}{L}\right)^{-\epsilon} \tag{62}$$

Now, we turn to the solution of $\dot{x}_L = 0$ and $\dot{z}_L = 0$. By replacing (58) and (59) in (54), we get the linear function $I_L \equiv I_L(x_L, x_H, z_L) = I_0 + I_1 x_H + I_2 x_L + I_3 z_L$, where $I_0 \equiv \frac{\Theta}{\zeta\lambda^{\frac{1-\alpha}{\alpha}}(lL)^{\epsilon}}(\chi_Y - \chi_X)\left[\left(\tilde{P}_H\right)^{\frac{1}{\alpha}} hH(\tilde{Q^*}) + \left(\tilde{P}_L\right)^{\frac{1}{\alpha}} lL\right]$, $I_1 \equiv -\frac{\Theta}{\lambda^{\frac{1-\alpha}{\alpha}}}\left(\frac{hH}{lL}\right)^{1-\epsilon}$, $I_2 \equiv -\frac{\Theta}{\lambda^{\frac{1-\alpha}{\alpha}}}$ and $I_3 \equiv -\frac{\Theta}{\zeta\lambda^{\frac{1-\alpha}{\alpha}}(lL)^{\epsilon}}$, and $\Theta \equiv \left[1+\left(\frac{hH}{lL}\right)^{1-\epsilon}\right]^{-1}$. Substituting in (57a) and (57b), equating $\dot{x}_L = 0$ and $\dot{z}_L = 0$ and solving for the interior equilibrium, yields

$$\tilde{z}_L = \left(-I_0 - I_1\tilde{x}_H - I_2\tilde{x}_L + \frac{\sigma+\gamma}{\Xi}\tilde{x}_L\right)\frac{1}{I_3} \tag{63}$$

$$\tilde{x}_L = \frac{\frac{\Xi}{\theta}\left[\frac{\pi}{\zeta}(lL)^{1-\epsilon}\left(\tilde{P}_L\right)^{\frac{1}{\alpha}} - \rho\right]}{\Xi(\sigma+\gamma+1) + \frac{1}{\theta}(\sigma+\gamma)} \tag{64}$$

Given that $x_L^{ss} = x_H^{ss}$ and (60), we can write

$$\tilde{x}_H = \frac{\frac{\Xi}{\theta}\left[\frac{\pi}{\zeta}(lL)^{1-\epsilon}\left(\tilde{P}_L\right)^{\frac{1}{\alpha}} - \rho\right]}{\Xi(\sigma+\gamma+1) + \frac{1}{\theta}(\sigma+\gamma)} = \frac{\frac{\Xi}{\theta}\left[\frac{\pi}{\zeta}(hH)^{1-\epsilon}\left(\tilde{P}_H\right)^{\frac{1}{\alpha}} - \rho\right]}{\Xi(\sigma+\gamma+1) + \frac{1}{\theta}(\sigma+\gamma)} \tag{65}$$

Also, from (50a) and (51)-(52), we find

$$\tilde{g}_{N_L} = \tilde{g}_{N_H} = \tilde{x}_L = \tilde{x}_H \tag{66}$$

and, from Proposition 1-(iv),

$$\tilde{g}_{Q_L} = \tilde{g}_{Q_H} = \tilde{g} = \frac{\frac{\Xi}{\theta}\left[\frac{\pi}{\zeta}(lL)^{1-\epsilon}\left(\tilde{P}_L\right)^{\frac{1}{\alpha}} - \rho\right](\sigma+\gamma+1)}{\Xi(\sigma+\gamma+1) + \frac{1}{\theta}(\sigma+\gamma)} \tag{67}$$

Finally, using (63) and the definition of I_0, I_1, I_2 and I_3, we have

$$\tilde{I}_L = \tilde{I}_H = \frac{\sigma+\gamma}{\Xi}\tilde{x}_L = \frac{\frac{1}{\theta}\left[\frac{\pi}{\zeta}(lL)^{1-\epsilon}\left(\tilde{P}_L\right)^{\frac{1}{\alpha}} - \rho\right](\sigma+\gamma)}{\Xi(\sigma+\gamma+1) + \frac{1}{\theta}(\sigma+\gamma)} \tag{68}$$

and

$$\tilde{z}_L = (\chi_Y - \chi_X)\left[\left(\tilde{P}_H\right)^{\frac{1}{\alpha}} hH(\tilde{Q^*}) + \left(\tilde{P}_L\right)^{\frac{1}{\alpha}} lL\right] -$$

$$- \left(\zeta\lambda^{\frac{1-\alpha}{\alpha}}\tilde{I}_L + \zeta\tilde{x}_L\right)\left[(hH)^{\epsilon}(\tilde{Q^*}) + (lL)^{\epsilon}\right] \quad (69)$$

The steady-state values of N_L, N_H and C are derived from (51)-(52) and (55), given Q_H (alternatively, Q_L). Thus,

$$\tilde{N}_L = \left(\frac{\zeta}{\phi}(lL)^{\epsilon}\right)^{\frac{1}{\sigma+\gamma+1}} (\tilde{x}_L)^{\frac{-\gamma}{\sigma+\gamma+1}} \left(\tilde{Q}_L\right)^{\frac{1}{\sigma+\gamma+1}} \quad (70a)$$

$$\tilde{N}_H = \left(\frac{\zeta}{\phi}(hH)^{\epsilon}\right)^{\frac{1}{\sigma+\gamma+1}} (\tilde{x}_H)^{\frac{-\gamma}{\sigma+\gamma+1}} \left(\tilde{Q}_H\right)^{\frac{1}{\sigma+\gamma+1}} \quad (70b)$$

$$\tilde{C} = \tilde{z}_L\tilde{Q}_L \quad (71)$$

We use (70a) and (70b) to derive

$$(\tilde{N^*}) \equiv \left(\frac{\tilde{N_H}}{N_L}\right) = \left(\frac{hH}{lL}\right)^{\frac{1-\epsilon}{\sigma+\gamma+1}} \quad (72)$$

Let the stock of technological-knowledge per firm, $\frac{Q_m}{N_m}$, be a measure of *average firm size*. The ratio between (72) and (59) yields the steady-state value for the following measure of *relative* average firm size

$$\left(\frac{\tilde{Q^*}}{N^*}\right) = \left(\frac{hH}{lL}\right)^{\frac{-\epsilon[1+2(\sigma+\gamma)]+\sigma+\gamma}{\sigma+\gamma+1}} \quad (73)$$

Alternative measures of firm size are production (or sales) per firm, $\frac{X_L}{N_L} = \chi_X P_L^{\frac{1}{\alpha}} lL\frac{Q_L}{N_L}$ (see (17) and (49); a similar expression obtains for $m = H$), and financial assets per firm, $\frac{a_L}{N_L} = \eta_L = \zeta\,(lL)^{\epsilon}\,\frac{Q_L}{N_L}$ (see (45) and (40); a similar expression obtains for $m = H$). Thus, we may consider

$$(\tilde{X^*}) \equiv \left(\frac{\tilde{X_H}}{X_L}\right) = \left(\frac{hH}{lL}\right)^{1-\epsilon} \quad (74)$$

$$\left(\frac{\tilde{X^*}}{N^*}\right) = \left(\frac{\tilde{P_H}}{P_L}\right)^{\frac{1}{\alpha}}\left(\frac{hH}{lL}\right)\left(\frac{\tilde{Q^*}}{N^*}\right) = \left(\frac{hH}{lL}\right)^{\frac{(1-\epsilon)(\sigma+\gamma)}{\sigma+\gamma+1}} \quad (75)$$

where (74) and (75) are derived from (17), (58), (59) and (72). Observe that, given the assumptions of our model,[20] we have also $\left(\frac{\tilde{a^*}}{N^*}\right) = \left(\frac{hH}{lL}\right)^{\epsilon}\left(\frac{\tilde{Q^*}}{N^*}\right) = \left(\frac{hH}{lL}\right)^{\frac{(1-\epsilon)(\sigma+\gamma)}{\sigma+\gamma+1}}$, where $a^* \equiv \frac{a_L}{a_H}$, that is, the alternative measures of *relative* firm size $\frac{X^*}{N^*}$ and $\frac{a^*}{N^*}$ coincide in steady state, whatever ϵ, in spite of the outstanding differences between the two

[20]Namely, we have $\zeta_L \equiv \zeta_H \equiv \zeta$.

alternative measures of *absolute* firm size, $\frac{a_m}{N_m}$ and $\frac{X_m}{N_m}$. Note also that $\frac{X^*}{N^*}$ and $\frac{a^*}{N^*}$ are strictly equal to $\frac{Q^*}{N^*}$ iff $\epsilon = 0$.

The results above can be summarized in the following proposition.

Proposition 2. There is a unique interior steady-state equilibrium, as defined by equations (59)-(71).

Proof: See derivations above.

Finally, $\tilde{g} > 0$ requires $\frac{\bar{\pi}}{\zeta}(lL)^{1-\epsilon}\left(\tilde{P}_L\right)^{\frac{1}{\alpha}} - \rho = \frac{\bar{\pi}}{\zeta}(hH)^{1-\epsilon}\left(\tilde{P}_H\right)^{\frac{1}{\alpha}} - \rho > 0$. Since, from (43), $g = g_C = \frac{1}{\theta}(r-\rho)$, then $r > \rho$ must occur. This condition also guarantees $\tilde{g}_{N_m} > 0$. On the other hand, according to the transversality condition, (44), together with (45) and (40), we have

$$\lim_{t\to\infty} e^{-\rho t}\cdot C(t)^{-\theta}\cdot\zeta\cdot(lL)^{\epsilon}\cdot Q_L(t) = \lim_{t\to\infty} e^{-\rho t}\cdot\left(\frac{C(t)}{Q_L(t)}\right)^{-\theta}\cdot\zeta\cdot(lL)^{\epsilon}\cdot Q_L(t)^{1-\theta} = 0 \quad (76)$$

where $\frac{C}{Q_L}$ is stationary in steady-state, as shown above. Let $Q_L = \hat{Q}_L e^{gt}$, where $\hat{Q}_L$ denotes detrended Q_L (thus stationary in steady-state), and substitute in (76), to see that the transversality condition implies $\rho \geq (1-\theta)g$; using again $g = \frac{1}{\theta}(r-\rho)$, the latter condition can be written alternatively as $r > g$. As it happens, this condition also guarantees that attainable utility is bounded, i.e., the integral (41) converges to infinity.

Thus, our model predicts, under a sufficiently productive technology, a steady-state equilibrium with constant positive g and g_N, where the former exceeds the latter by an amount corresponding to the growth of intermediate-good quality, driven by vertical innovation; to verify this, just replace (50a) in (50b) and solve to get $\frac{\dot{Q}_m}{Q_m} - \frac{\dot{N}_m}{N_m} = I_m\cdot\left(\lambda^{\frac{1-\alpha}{\alpha}} - 1\right)$. This implies that the consumption growth rate equals the growth rate of the number of varieties plus the growth rate of intermediate-good quality, in line with the view that industrial growth proceeds both along an intensive and an extensive margin.Similarly to Gil, Brito, and Afonso (2008), variety expansion is sustained by endogenous technological-knowledge accumulation (independently of population growth), as the expected growth of intermediate-good quality due to vertical R&D makes it attractive, in terms of intertemporal profits, for potential entrants to always put up an entry cost, in spite of its increase with N_m.[21]

4. Comparative steady-state results

Now, we discuss the comparative statics of the interior steady-state,[22] namely concerning the impact of changes in $\frac{hH}{lL}$, ϵ, σ and γ on the number of firms, technology-knowledge

[21] Observe also that $\tilde{P}_L$, $\tilde{P}_H$ may differ from unity (see (60)). If $\tilde{P}_m$ exceeds (is less than) unity, the production technology can (must) be proportionaly less (more) productive than in the single-technology model (where $P_L \equiv P_H \equiv P_Y = 1$) in order to ensure $\tilde{g} > 0$.

[22] Henceforth, the ~ is omitted for the sake of simplicity.

stock, production (sales) and the average firm size in the H-complementary-technology vis-à-vis the L-complementary-technology sector (henceforth, we refer to the latter as sector H vis-à-vis sector L). That is, we focus on N^*, Q^*, X^*, $\frac{Q^*}{N^*}$ and $\frac{X^*}{N^*}$.[23]

4.1. "Cross-country" analysis

4.1.1. Labour endowment, scale effects and industrial structure

The proposition below summarises the main results with respect to the impact of changes in $\frac{hH}{lL}$ on the industrial structure, as characterised by the number of firms and average firm size across sectors, considering four critical values for the degree of scale-effects removal, ϵ. One can interpret this exercise as a *cross-section* comparison of industrial structures between countries with different levels of $\frac{hH}{lL}$.

Proposition 3 The higher $\frac{hH}{lL}$: (i) if $\epsilon = 0$, the higher N^*, Q^*, $\frac{Q^*}{N^*}$, X^* and $\frac{X^*}{N^*}$; (ii) if $\epsilon = \bar{\epsilon} \equiv \frac{\sigma+\gamma}{1+2(\sigma+\gamma)}$, the higher N^*, Q^*, X^* and $\frac{X^*}{N^*}$ with an invariant $\frac{Q^*}{N^*}$; (iii) if $\epsilon = \frac{1}{2}$, the higher N^*, X^* and $\frac{X^*}{N^*}$, and the lower $\frac{Q^*}{N^*}$, with an invariant Q^*; (iv) if $\epsilon = 1$, the lower $Q^*$and $\frac{Q^*}{N^*}$, with invariant N^*, $X^*$and $\frac{X^*}{N^*}$.

Proof: Differentiate (59), (72), (73) and (75) with respect to $\frac{hH}{lL}$.

The critical value $\epsilon = 0$ corresponds to the case of complete (positive) scale effects (e.g., Acemoglu and Zilibotti, 2001), whereas $\epsilon = 1$ corresponds to the case of no scale effects (e.g., Afonso, 2006). The intermediate critical values correspond to the case where scale effects, although existent, have either no impact on $\frac{Q^*}{N^*}$ ($\epsilon = \bar{\epsilon}$) or on $Q^*(\epsilon = \frac{1}{2})$.

We look at the mechanism that links relative labour endowment to industrial structure by focusing on three intervals for ϵ. When $0 \leq \epsilon < \bar{\epsilon}$, i.e., the case of *large positive scale effects*, a country with a higher $\frac{hH}{lL}$ is expected to have a *largernumber of firms and a larger average firm size in sector H vis-à-vis sector L*. In fact, a higher relative endowment of skilled labour induces a larger number of firms *and* aggregate technological-knowledge stock, in relative terms, in the sector that produces skilled-labour complementary goods; however, the increment obtained in the technological-knowledge stock is larger than in the number of firms, therefore implying an also larger average firm size in sector H vis-à-vis sector L. When $\epsilon > 1$, i.e., the case of *negative scale effects*, the positive relationship between the number of firms and average firm size is also verified but, unsurprisingly, larger values obtain with a *smaller* endowment of skilled labour vis-à-vis unskilled labour. Finally, when $\bar{\epsilon} < \epsilon < 1$, i.e., the case of *positive but small scale effects*, a country with a higher $\frac{hH}{lL}$ is expected to have a *larger number of firms but a smaller average firm size in sector H vis-à-vis sector L*.In particular, if $\bar{\epsilon} < \epsilon < \frac{1}{2}$, Q^* increases less that N^* in response to a higher $\frac{hH}{lL}$, whereas if $\frac{1}{2} < \epsilon < 1$, Q^* falls and N^* increases in response to a higher $\frac{hH}{lL}$.[24]

[23] The comparative statics concerning the remaining structural parameters and endogenous variables can be found in Gil, Brito, and Afonso (2008).

[24] Observe that $\bar{\epsilon} \equiv \frac{\sigma+\gamma}{1+2(\sigma+\gamma)} < \frac{1}{2}$, whatever σ and γ positive and finite.

The results above stem from the different response of Q^* and N^* to $\frac{hH}{lL}$ (see (59) and (72)). In turn, this mirrors the dominant impact of scale effects on Q^* through the vertical-innovation mechanism, impacting only through the latter on the horizontal-innovation mechanism, thereby affecting N^*,[25]combined with the fact that only the horizontal entry dynamics is regulated by variable entry costs with elasticities σ and γ. The higher sensitivity of Q^* to scale effects is clear from Proposition 1, according to which Q^* is constant with respect to $\frac{hH}{lL}$ when $\epsilon = \frac{1}{2}$ (i.e., 50 percent of scale effects removed), while N^* is constant with respect to $\frac{hH}{lL}$ when $\epsilon = 1$ (i.e., complete removal of scale effects). This also explains the effect of σ and γ on the value of the endogenous threshold $\bar{\epsilon}$: the lower σ and γ, the lower $\bar{\epsilon}$, i.e., the higher the degree of scale effects below which there is a falling relative firm size with $\frac{hH}{lL}$.

We also draw our attention to the alternative measure of relative firm size $\frac{X^*}{N^*}$ (or $\frac{a^*}{N^*}$). In this case, the model predicts that, if $0 \leq \epsilon < 1$ (respectively, $\epsilon > 1$), a higher (lower) $\frac{hH}{lL}$ always corresponds to a *higher average firm size and ahigher number of firms in sector H vis-à-vis sector L*.The behaviour of $\frac{X^*}{N^*}$ is explained by the autonomous effect of $\frac{hH}{lL}$, through the price ratio $\frac{P_H}{P_L}$, on $\frac{X^*}{N^*}$, in addition to the basic channel that impacts on $\frac{Q^*}{N^*}$. In Appendix D, Figure 2 depicts the relationship between $\frac{hH}{lL}$ and $\frac{Q^*}{N^*}$, $\frac{X^*}{N^*}$, and N^*.

Cross-country evidence in Figure 1, Appendix A, suggests that the relative number of firms tends to be (i) positively correlated with relative production, but (ii) (slightly) negatively correlated with relative average firm size. Relationship (i) is compatible with our results with respect to N^* and X^*, provided there are some scale effects (positive or negative), as well as to N^* and Q^*, if scale effects are either negative or positive and large; however, relationship (ii) is in line with our results only if we consider relative firm size measured as $\frac{Q^*}{N^*}$, in the case of small positive scale effects. As explained above, if $\bar{\epsilon} < \epsilon < \frac{1}{2}$, both (i) and (ii) are compatible with the predictions of our model.

4.1.2. Labour endowment, scale effects and the concentration index

In the tradition of the IO literature, we are also interested in the relation between $\frac{hH}{lL}$, ϵ and a synthetic measure of industry concentration. Concentration measures widely used in the literature are, e.g., the k-firm concentration ratio (the sum of the market share of the k^{th} biggest firms in a given industry) and the Herfindahl index (the quadratic sum of the shares of all firms in a given industry).

However, one must take into account two particular features of our model when choosing which concentration measure to adopt. First, the relevant "market" to measure the degree of concentration is established at the economy-wide level, since the economy comprises a continuum of monopolistic industries producing imperfectly substitutable goods. Second, the "equilibrium" values of the endogenous variables of interest (namely, output

[25]Notice that, given the postulated horizontal entry technology and our lab-equipment specification, the vertical-innovation mechanism ultimately commands the horizontal entry dynamics in steady state, meaning that a steady state with increasingly costly entry only occurs because entrants expect incumbency value to grow propelled by quality-enhancing R&D.

and the number of firms) are non-stationary, in the sense that they are defined along a balanced-growth path characterised by positive growth rates. Thus, by construction, the k-firm concentration ratio is devoided of interest in the light of the first feature, whereas the Herfindahl index is not suitable specially given the second one (it tends to zero as N_m grows at the rate $g_N > 0$ along the balanced-growth path).[26] A similar appreciation can be done with respect to other commonly used concentration measures.

Thus, we build our own synthetic measure of concentration, adapted to the context of our model. We propose a two-dimension aggregate concentration index with two versions, by combining the "market share" of the number of firms, $\frac{N_m}{N}$, with the "market share" of either the technological-knowledge stock, $\frac{Q_m}{Q}$, or the output (sales), $\frac{X_m}{X}$, of each technological sector ($m = H, L$), as follows

$$\Sigma_Q \equiv \frac{Q_L}{Q} \cdot \frac{N_L}{N} + \frac{Q_H}{Q} \cdot \frac{N_H}{N} = \frac{Q^* \cdot N^* + 1}{Q^* \cdot N^* + Q^* + N^* + 1} \tag{77a}$$

$$\Sigma_X \equiv \frac{X_L}{X} \cdot \frac{N_L}{N} + \frac{X_H}{X} \cdot \frac{N_H}{N} = \frac{X^* \cdot N^* + 1}{X^* \cdot N^* + X^* + N^* + 1} \tag{77b}$$

Our concentration index compares sectors at the economy-wide level instead of individual firms within a sector and is constant in steady state (see (59), (72) and (74)). The aggregate character of our index accommodates the view, noted by Kamien and Schwatrz (1975), that traditional industry boundaries seem to become decreasingly useful for the purpose of economic analysis "as new products and processes compete *across* industry lines", which is a salient feature of our model.

The propositions below characterise the behaviour of the two versions of our concentration index with respect to $\frac{hH}{lL}$ and ϵ.

Proposition 4.1 (i) If $0 \leq \epsilon < \frac{1}{2}$ or $\epsilon > 1$, and $\frac{hH}{lL} > 1 \wedge \frac{hH}{lL} \to \infty$ or $\frac{hH}{lL} < 1 \wedge \frac{hH}{lL} \to 0$, then $\Sigma_Q \to 1$; (ii) if $\frac{1}{2} < \epsilon < 1$, and $\frac{hH}{lL} > 1 \wedge \frac{hH}{lL} \to \infty$ or $\frac{hH}{lL} < 1 \wedge \frac{hH}{lL} \to 0$, then $\Sigma_Q \to 0$; (iii.a) if $\frac{hH}{lL} = 1$, whatever $\epsilon > 0$, then $\Sigma_Q = \frac{1}{2}$; (iii.b) if $\epsilon = \frac{1}{2}$ or $\epsilon = 1$, and $\frac{hH}{lL} > 1 \wedge \frac{hH}{lL} \to \infty$ or $\frac{hH}{lL} < 1 \wedge \frac{hH}{lL} \to 0$, then $\Sigma_Q \to \frac{1}{2}$.

Proposition 4.2 (i) If $0 \leq \epsilon < 1$ or $\epsilon > 1$, and $\frac{hH}{lL} > 1 \wedge \frac{hH}{lL} \to \infty$ or $\frac{hH}{lL} < 1 \wedge \frac{hH}{lL} \to 0$, then $\Sigma_X \to 1$; (ii) if $\frac{hH}{lL} = 1$, whatever $\epsilon > 0$, or if $\epsilon = 1$, whatever $\frac{hH}{lL} > 0$, then $\Sigma_X = \frac{1}{2}$.

Proof: See Appendix E.

Thus, $0 \leq \Sigma_Q \leq 1$ but $\frac{1}{2} \leq \Sigma_X \leq 1$. As shown in detail in Appendix E, the boundaries to the concentration index are to be interpreted as follows: $\Sigma_Q = 1$ refers to *total concentration of firms and technological-knowledge stock in a single technological sector* and $\Sigma_Q = 0$ refers to *total concentration of firms in one technological sector and of technological-knowledge stock in the other* . We identify the extreme case $\Sigma_Q = 1$ as industry concentration of type I and $\Sigma_Q = 0$ as industry concentration of type II. In

[26] Observe that the version of the concentration ratio usually applied to symmetric equilibria, $\frac{1}{N_m}$, also tends to zero along the balanced growth path.

contrast, $\Sigma_Q = 0.5$ implies, in general, an *uniform distribution of firms and technological-knowledge stock across sectors.*[27]

The relationship between N^* and X^* determines, by construction, that the lower boundary to Σ_X is $\frac{1}{2}$ (*uniform distribution of firms and output across sectors*), such that this version of the concentration index only admits industry concentration of type I (*total concentration of firms and output in one technological sector*).

By interpreting this exercise again as a cross-country comparison of industrial structures, we conclude that, depending on ϵ, the industrial structure of a given country is characterised by either $\frac{1}{2} \leq \Sigma_Q \leq 1$ or $0 \leq \Sigma_Q \leq \frac{1}{2}$. When $\epsilon > 1$ or $0 \leq \epsilon < \frac{1}{2}$, i.e., the case of either negative or large positive scale effects, a country with a *higher* $\frac{hH}{lL} > 1$ is expected to have a higher concentration index, towards $\Sigma_Q = 1$, i.e., *concentration of type I*; a country with a *higher* $\frac{hH}{lL} < 1$ is expected to have a lower concentration index, towards $\Sigma_Q = \frac{1}{2}$, i.e., *an uniform distribution.* Thus, Σ_Q *displays a U-shaped behaviour with respect to* $\frac{hH}{lL}$, with its minimum at $\frac{hH}{lL} = 1$.

When $\frac{1}{2} < \epsilon < 1$, i.e., the case of positive but small scale effects, a country with a *higher* $\frac{hH}{lL} > 1$ is expected to have a lower concentration index, towards $\Sigma_Q = 0$, i.e., *concentration of type II*; a country with a *higher* $\frac{hH}{lL} < 1$ is expected to have a higher concentration index, towards $\Sigma_Q = \frac{1}{2}$, i.e., *an uniform distribution.* Thus, Σ_Q *displays an inverted U-shaped behaviour with respect to* $\frac{hH}{lL}$, with its maximum at $\frac{hH}{lL} = 1$.[28]

In contrast, the model predicts a *U-shaped behaviour of* Σ_X *with respect to* $\frac{hH}{lL}$,with its minimum at $\frac{hH}{lL} = 1$, whatever $\epsilon \neq 1$, i.e., provided there are some scale effects (whether positive or negative). Figure 3.2, Appendix E, depicts the cases of non-monotonic relationship between the concentration index and $\frac{hH}{lL}$.

Computation of the concentration index by using the data in Figure 1, Appendix A, shows that the index displays values above 0.5, hence implying industry concentration of type I. This is compatible with our results if we consider Σ_X, provided there are some scale effects (positive or negative), or Σ_Q, in the case of negative or large positive scale effects. On the other hand, the concentration index and both relative production and the relative number of firms display a negative correlation across countries, which is also compatible with our model as follows: if $N^* < 1$, $X^* < 1$ reflect $\frac{hH}{lL} < 1$, then higher N^*, X^*, due to higher $\frac{hH}{lL}$, must imply a lower concentration index, down from 1 towards 0.5.

[27]In rigour, this corresponds only to (iii.a) in Proposition 4.1. We interpret (iii.b) as a "degenerate" case associated with specific levels of scale effects ($\epsilon = \frac{1}{2}$ and $\epsilon = 1$), as explained in Appendix E.

[28]We note that, here, the the critical value that separates the "large" from the "positive but small scale effects" is $\epsilon = \frac{1}{2}$, reflecting the fact that Q^* is constant with respect to $\frac{hH}{lL}$ when ϵ takes that value. In Subsection 4.1.1, above, the relevant critical value is $\epsilon = \bar{\epsilon}$, since therein we are interested in the behaviour of $\frac{Q^*}{N^*}$, which is constant with respect to $\frac{hH}{lL}$ when ϵ takes the latter value.

4.2. "Intra-country" analysis

4.2.1. Scale effects and industrial structure

In this subsection, we focus on the comparison between the relative number of firms and the relative average firm size for a given $\frac{hH}{lL}$, in order to conclude on the skewness of the steady-state distribution of average firm size between sectors L and H in an *intra-country* perspective.[29]The industrial structure is characterised by a skewed distribution as follows

Proposition 5.1 (i) If $\bar{\epsilon} < \epsilon < 1$, then $N^* > (<)1$ and $\frac{Q^*}{N^*} < (>)1$, for a given $\frac{hH}{lL} > (<)1$, thus corresponding to a *right-skewed average size distribution* ("more small-average firms than large-average firms"); and (ii) if $0 \leq \epsilon < \bar{\epsilon}$ [respectively, $\epsilon > 1$], then $N^* > (<)1 \wedge \frac{Q^*}{N^*} > (<)1$, for a given $\frac{hH}{lL} > (<)1$ [$\frac{hH}{lL} < (>)1$], thus corresponding to a *left-skewed average size distribution* ("more large-average firms than small-average firms").

Proposition 5.2 If $0 \leq \epsilon < 1$ [respectively, $\epsilon > 1$], then $N^* > (<)1 \wedge \frac{X^*}{N^*} > (<)1$, for a given $\frac{hH}{lL} > (<)1$ [$\frac{hH}{lL} < (>)1$], thus corresponding to a *left-skewed average-firm-size distribution.*

Proof: Immediate by inspection of (72), (73) and (75).

From Proposition 4, we learn that only in the case of *positive but small scale effects,* i.e., $\bar{\epsilon} < \epsilon < 1$, and $\frac{hH}{lL} \neq 1$ does the number of firms in the sector with smaller average firm size exceeds the number in the sector with larger average firm size, thus obtaining a skewness which is in line with the IO stylised facts on the size distribution of *individual* firms (see, e.g., Sutton, 1997; and Cabral and Mata, 2003). The mechanism behind this result is the same as the one described in Proposition 3, above. In the particular case of $\epsilon = \bar{\epsilon}$ and $\frac{hH}{lL} \neq 1$, the two sectors have a different number of firms, but their average size is the same; if $\epsilon = 1$ and $\frac{hH}{lL} \neq 1$, the opposite is true, with the two sectors exhibiting the same number of firms, but with different average size. When $\frac{hH}{lL} = 1$, the distribution is uniform between sectors (same number of firms and same average firm size), whatever $\epsilon > 0$.

However, if we consider the alternative measure of relative firm size, $\frac{X^*}{N^*}$(see (75)), then the model always predicts a left-skewed average-firm-size distribution, provided $\frac{hH}{lL} \neq 1$ and $\epsilon \neq 1$. If $\frac{hH}{lL} = 1$, whatever $\epsilon > 0$, or if $\epsilon = 1$, whatever $\frac{hH}{lL} > 0$, the distribution is uniform between sectors. Figure 2, Appendix D, makes clear the differences between $\frac{Q^*}{N^*}$ and $\frac{X^*}{N^*}$, for given ϵ and $\frac{hH}{lL}$.

Now, let us take heterogenous horizontal flow fixed entry costs ($\phi_L \neq \phi_H$) into account, such that (72), (73) and (75) are rewritten as $N^* = \left(\frac{\phi_L}{\phi_H}\right)^{\frac{1}{\sigma+\gamma+1}} \cdot \left(\frac{hH}{lL}\right)^{\frac{1-\epsilon}{\sigma+\gamma+1}}$, $\frac{Q^*}{N^*} = \left(\frac{\phi_H}{\phi_L}\right)^{\frac{1}{\sigma+\gamma+1}} \cdot \left(\frac{hH}{lL}\right)^{\frac{-\epsilon[1+2(\sigma+\gamma)]+\sigma+\gamma}{\sigma+\gamma+1}}$ and $\frac{X^*}{N^*} = \left(\frac{\phi_H}{\phi_L}\right)^{\frac{1}{\sigma+\gamma+1}} \cdot \left(\frac{hH}{lL}\right)^{\frac{(1-\epsilon)(\sigma+\gamma)}{\sigma+\gamma+1}}$. We find that

[29]Our model could be extended to a setting of multiple types of human capital and, thus, multiple complementary-technology sectors, in order to generate a smoother size distribution.

our initial results of left-skewness are overturned for a value of $\frac{\phi_H}{\phi_L}$ sufficiently above (or sufficiently below) unity, such that, e.g., one gets either $N^* > 1 \wedge \frac{X^*}{N^*} < 1 (\Leftarrow \phi_L > \phi_H)$ or $N^* < 1 \wedge \frac{X^*}{N^*} > 1$ $(\Leftarrow \phi_L < \phi_H)$. Observe that ϕ_m only impacts on $\frac{X^*}{N^*}$ and $\frac{Q^*}{N^*}$ through N^* (see (70a) and (70b)),[30] as there is no relationship between Q^* (and X^* through Q^*) and ϕ_m (see (57d)-(59)). This results from the dominant effect exherted by the vertical-innovation mechanism over the horizontal entry dynamics, already explained above.[31,32]

But, irrespective of the consideration of heterogeneous entry costs, we note that the empirical regularity of right-skewed individual-firm-size distribution and our theoretical result of left-skewed average-firm-size distribution are not necessarily contraditory. In fact, by pooling individual firms (selected from a given right-skewed size distribution) in complementary-technology sectors, one may be able to observe empirically that the number of firms in the sector with larger average firm size exceeds the number in the sector with smaller average firm size. In particular, this can be shown to be true as long as the subsets of firms in each sector are characterised by right-skewed size distributions whose mean and variance differ by the necessary ammount.[33]

According to the data in Figure 1, Appendix A, the relative number of firms and relative production display values below unity. This is in line with our results with respect to X^* (respectively, Q^*) given $\frac{hH}{lL} < 1$, if scale effects are positive (large positive), or $\frac{hH}{lL} > 1$, if scale effects are negative. On the other hand, data shows relative firm size systematically above unity. This is compatible, in theoretical terms, with the observed relative number of firms below unity - implying a right-skewed firm size distribution - (i) if we consider relative firm size measured as $\frac{Q^*}{N^*}$, in the case of small positive scale effects, or (ii) heterogenous horizontal flow fixed entry costs of sufficient size combined with relative firm size measured as either $\frac{X^*}{N^*}$, provided there are some scale effects (positive or negative), or $\frac{Q^*}{N^*}$, in the case of negative or large positive scale effects.

Finally, we are interested in analysing the impact of changes in the dynamic entry costs, represented by the elasticities of the horizontal-entry cost function, σ and γ, in the *asymmetry* of the industrial structure, by keeping $\frac{hH}{lL}$ and ϵ as given. The following proposition summarises the main results

Proposition 6 When σ and γ increase: (i) if $\frac{hH}{lL} > (<) 1$, then N^* *decreases*, and $\frac{X^*}{N^*}$ and $\frac{Q^*}{N^*}$ *increase*, whatever $0 \leq \epsilon < 1$ $(\epsilon > 1)$; (ii) if $\frac{hH}{lL} < (>) 1$, then N^* *increases,*

[30] Comparing with the IO literature that focus on the influence of entry costs on firm dynamics and firm size distribution, we note that the role of (flow fixed) entry costs in the derivation of the average-firm-size distribution in our model is confined to the *static* determination of the relative number of firms in each technological group. In contrast, the referred literature focus on the *dynamic* effects of (sunk) costs, i.e., the effects on entry dynamics and post-entry firm growth, thereby deriving results on the ergodic individual-firm-size distribution (e.g., Hopenhayn, 1992; and Ericson and Pakes, 1995).

[31] The same mechanism explains why ϕ_m has no effect on g (see (67)).

[32] The reversal of the left-skewness result could also be obtained by considering $\zeta_L \neq \zeta_H$. However, ζ_m has a direct effect on both Q^* (through P^*; see (57d)-(59)), and N^* (see (70a) and (70b)); ζ_m has also an impact on g.

[33] Have in mind that, given the Poisson-driven quality-ladders mechanism that characterises firms in each technological sector in our model, it is possible to derive, under certain standard conditions, an asymptotic right-skewed firm size distribution (the lognormal distribution) for each sector (see Segerstrom, 2007).

and $\frac{X^*}{N^*}$ and $\frac{Q^*}{N^*}$ *decrease*, whatever $0 \leq \epsilon < 1$ ($\epsilon > 1$); (iii) if $\frac{hH}{lL} = 1$ or $\epsilon = 1$ (or $\epsilon = \bar{\epsilon}$, for $\frac{Q^*}{N^*}$), then N^*, $\frac{X^*}{N^*}$ and $\frac{Q^*}{N^*}$ are independent of σ and γ.

Proof: Differentiate (72), (73) and (75) with respect to σ and γ.

In order to interpret the last proposition in terms of the influence of σ and γ in the asymmetry of the industrial structure, one must consider whether the initial values of N^*, $\frac{X^*}{N^*}$ and $\frac{Q^*}{N^*}$ are above or below unity. Hence, have in mind that $\frac{hH}{lL} > (<)1 \Rightarrow N^* > 1$, $\frac{X^*}{N^*} > 1$ and $\frac{hH}{lL} < (>)1 \Rightarrow N^* < 1$, $\frac{X^*}{N^*} < 1$, for a given $0 \leq \epsilon < 1$ ($\epsilon > 1$); on the other hand, $\frac{hH}{lL} > (<)1 \Rightarrow \frac{Q^*}{N^*} > 1$, if $0 \leq \epsilon < \bar{\epsilon}$ ($\epsilon > 1$), and $\frac{Q^*}{N^*} < 1$, if $\bar{\epsilon} < \epsilon < 1$, whereas $\frac{hH}{lL} < (>)1 \Rightarrow \frac{Q^*}{N^*} < 1$, if $0 \leq \epsilon < \bar{\epsilon}$ ($\epsilon > 1$), and $\frac{Q^*}{N^*} > 1$, if $\bar{\epsilon} < \epsilon < 1$.

Thus, according to Proposition 5, an increase in the dynamic entry costs leads to a decrease, in relative terms, in the number of firms in the sector that has more of them (in sector H vis-à-vis sector L, if $N^* > 1$, and vice-versa, if $N^* < 1$), whatever $\epsilon \neq 1$, i.e., dynamic entry costs have always a *stabilising effect* with respect to the number of firms, provided there are some scale effects (whether positive or negative). In turn, this impacts with opposite sign on firm size, either measured as $\frac{X}{N}$ or $\frac{Q}{N}$, since σ and γ have no direct effect on Q^* or X^*(see (59) and (74)). When firm size is measured as $\frac{X}{N}$, an increase in σ and γ leads to an increase, in relative terms, in the average firm size in the sector that has larger firms, whatever $\epsilon \neq 1$, i.e., *dynamic entry costs have always a destabilising effect with respect to average firm size, provided there are some scale effects, thus contervailing the stabilizing effect on the number of firms.*

However, when firm size is measured as $\frac{Q}{N}$, an increase in σ and γ have *a destabilising effect* with respect to average firm size only when scale effects are either negative or positive and large (i.e., $\epsilon > 1$ or $0 \leq \epsilon < \bar{\epsilon}$). When scale effects are positive but small ($\bar{\epsilon} < \epsilon < 1$), an increase in σ and γ leads to an increase, in relative terms, in the average firm size in the sector that has smaller firms, i.e., dynamic entry costs have a *stabilising effect* with respect to average firm size. Given the described effect on N^*, we then find that *dynamic entry costs have a global stabilising effect on industrial structure only whenscale effects are positive but small. Otherwise, the impact on the number of firms contervails the effect on average firm size.*

4.2.2. Dynamic entry costs, the concentration index and aggregate growth

In this subsection, we explore the association between industrial concentration, measured by our concentration index, and aggregate long-term growth, g, by taking into account the simultaneous impact of changes in the dynamic entry costs, σ and γ, on both variables.[34] First, we analyse the effect of changes in σ and γ on g.

Proposition 7 The aggregate growth rate, g, is *decreasing* in the elasticities σ and γ.

Proof: Differentiate (66) with respect to σ and γ.

[34] Note that σ and γ are the only primitive parameters of the model that influence simultaneously g, Σ_Q and Σ_X, for a given set (ϵ, $\frac{hH}{lL}$) (see (67), (59), (72) and (74)).

The next propositions summarise the effect of σ and γ on Σ_Q and Σ_X.

Proposition 8.1 (i) If $0 \leq \epsilon < \frac{1}{2}$ or $\epsilon > 1$, and $\frac{hH}{lL} \neq 1$, then Σ_Q is *decreasing* in σ and γ (towards $\Sigma_Q = \frac{1}{2}$); (ii) if $\frac{1}{2} < \epsilon < 1$, and $\frac{hH}{lL} \neq 1$, then Σ_Q is *increasing* in σ and γ (towards $\Sigma_Q = \frac{1}{2}$); (iii) if $\epsilon = 1$ or $\frac{hH}{lL} = 1$, then Σ_Q is independent of σ and γ.

Proposition 8.2 (i) If $0 \leq \epsilon < 1$ or $\epsilon > 1$, and $\frac{hH}{lL} \neq 1$, then Σ_X is *decreasing* in σ and γ (towards $\Sigma_X = \frac{1}{2}$); (ii) if $\epsilon = 1$ or $\frac{hH}{lL} = 1$, then Σ_X is independent of σ and γ.

Proof: Differentiate (77a) and (77b) with respect to σ and γ.

From Propositions 7 and 8, we conclude that, with respect to Σ_Q, there is a *positive relationship between aggregate growth and concentration of type I*, when scale effects are negative or positive and large, but a *positive relationship between aggregate growth and concentration of type II*, when scale effects are positive but small. With respect to Σ_X, there is *a positive relationship between aggregate growth and concentration of type I*, provided there are at least some scale effects, whether positive ornegative. By joining Proposition 6, 7 and 8, we learn that a higher aggregate growth is expected to come hand-in-hand with a higher concentration of type I (respectively, type II), but also with a less (more) *asymmetric* distribution of firm size across sectors.[35]

According to the data in Figure 1, Appendix A, the concentration index and aggregate per capita growth display a positive correlation, which is in line with our model if we consider Σ_X, whatever the level of (non-null) scale effects, or with Σ_Q, in the case of either negative or large positive scale effects.

It is clear from the results above how the general equilibrium nature of our model allows for simultaneous determination of aggregate growth and industrial structure - in contrast to the traditional focus on causal effects -, as a subset of the technology parameters of the model (σ, γ) that determine the aggregate growth rate also influence concentration.

Finally, it should also be clear that our results with respect to the association between aggregate growth and industrial structure concern its *quantitative* dimension (i.e., how many firms and how much production are allocated to each sector vis-à-vis the others) and not its *qualitative* dimension (i.e., concentration of economic activity in a specific type of sector), as pursued by the literature of structural change (e.g., Fagerberg, 2000 and Bonatti and Felice, 2008).

4.2.3. Schumpeterian hypothesis

A wide range of empirical interpretations of the Schumpeterian hypothesis has led to a diversity of tests, in particular, involving the relationship between R&D activity and both market power and firm size, each variously measured (Kamien and Schwatrz, 1975). Less frequently recognised are the feedbacks between the latter variables and R&D activity.

[35] In rigour, since $\bar{\epsilon} < \frac{1}{2}$ for σ and γ positive and finite, one must also take into account the interval $\epsilon \in (\bar{\epsilon}, \frac{1}{2})$, for which a higher concentration of type I relates to a more asymmetric distribution of firm size.

On the other hand, data availability has allowed more extensive investigation of the relation of R&D activity with firm size than with market power. Therefore, there is a large literature studying whether R&D activity increases more than proportionaly with firm size, i.e., whether large firms are more R&D-intensive than small firms. At least among R&D-reporting firms, the evidence suggests that R&D increases in proportion to sales, poiting to R&D intensity independent of firm size (Klette and Kortum, 2004).

Having the above in mind, we now explore a version of the Schumpeterian hypothesis by studying the interrelation between R&D intensity, measured as the ratio between R&D outlays and firm size, and average firm size, measured as $\frac{Q}{N}$ or $\frac{X}{N}$. As referred before, in our model there is no causality to be imputed to these relationships, since average firm size and R&D activityare simultaneously determined.

Firstly, use (32), (40) and (48) to get $R_v^* \equiv \frac{R_{vH}}{R_{vL}} = \left(\frac{hH}{lL}\right)^{\epsilon} Q^*$ and $R_n^* \equiv \frac{R_{nH}}{R_{nL}} = \left(\frac{hH}{lL}\right)^{\epsilon} Q^*$. Then, combine the latter with (59) and (74) to find the R&D-intensity ratios

$$\frac{R^*}{Q^*} = \left(\frac{hH}{lL}\right)^{\epsilon} \tag{78a}$$

$$\frac{R^*}{X^*} = 1 \tag{78b}$$

where $R^* \equiv R_n^* = R_v^*$. The relation between R&D intensity and average firm size is characterised as follows

Proposition 9.1 (i) If $\bar{\epsilon} < \epsilon < 1$, then $\frac{R^*}{Q^*} > (<)1$ and $\frac{Q^*}{N^*} < (>)1$, for a given $\frac{hH}{lL} > (<)1$; and (ii) if $0 \leq \epsilon < \bar{\epsilon}$ [respectively, $\epsilon > 1$], then $\frac{R^*}{Q^*} > (<)1 \wedge \frac{Q^*}{N^*} > (<)1$, for a given $\frac{hH}{lL} > (<)1$ [$\frac{hH}{lL} < (>)1$].

Proposition 9.2 If $0 \leq \epsilon < 1$ [respectively, $\epsilon > 1$], then $\frac{R^*}{X^*} = 1 \wedge \frac{X^*}{N^*} > (<)1$, for a given $\frac{hH}{lL} > (<)1$ [$\frac{hH}{lL} < (>)1$].

Proof: Immediate by inspection of (73), (78a), (75) and (78b).

According to Proposition 9, the version of the Schumpeterian hypothesis that relates positively R&D intensity and firm size is only predicted by our model when size is measured as $\frac{Q}{N}$ and scale effects are either negative or positive and large. In contrast, if scale effects are positive but small, *the technological sector with smaller average firm size exhibits higher levels of R&D intensity*. One must also consider the following cases: if $\frac{hH}{lL} = 1$, whatever $\epsilon > 0$, R&D intensity and average firm size are both constant across sectors; if $\epsilon = 0$, whatever $\frac{hH}{lL} > 0$, R&D intensity is unchanged across sectors, irrespective of the behaviour of firm size; and if $\epsilon = \bar{\epsilon}$, firm size is constant across sectors, irrespective of the behaviour of R&D intensity. Thus, in the latter two particular cases, the model implies that *R&D intensity is independent of firm size*, which is one of the stylized facts that have emerged from firm-level empirical studies (Klette and Kortum, 2004).

Also important, when firm size is measured as $\frac{X}{N}$, R&D intensity is unchanged across sectors, irrespective of the behaviour of firm size, whatever $\epsilon \neq 1$. Thus, the model again implies that *R&D intensity is independent of firm size*; however, with R&D intensity measured as a ratio to sales, it is just sufficient to have some degree of scale effects, whether positive or negative, to obtain this result.[36]

The data in Figure 1, Appendix A, shows relative R&D intensity above unity, which is compatible, in theoretical terms, with the observed relative firm size also above unity if we consider relative R&D intensity measured as $\frac{R^*}{Q^*}$ and hence the relative firm size measured as $\frac{Q^*}{N^*}$ (i) in the case of either negative or large positive scale effects, or (ii) if we combine $\frac{Q^*}{N^*}$ with heterogenous horizontal flow fixed entry costs of sufficient size, also in the case of small positive scale effects (see Subsection 4.2.1).[37]In any case, the data we present seems to corroborate the Schumpeterian hypothesis as defined at the technological-sector level, in contrast to the conclusions from firm-level empirical evidence, referred to above.

5. Concluding remarks

This paper studies a specific dimension of the industrial structure, that of the distribution of firms and production across technological sectors, by building an endogenous-growth model of directed technical change that merges the expanding-variety with the quality-ladders mechanism. Our general equilibrium framework allows us to accommodate the view that the relationship between industrial structure, innovative activity and aggregate growth is not a causal one, but instead that they are simultaneously (endogenously) determined.

The model presented herein provides one possible economic mechanism to explain the data on the number of firms, production, average firm size and R&D intensity in high-tech vis-à-vis low-tech sectors in a set of european countries. It also provides a theoretical instrument to study the association between concentration, measured at the aggregate level, and long-run aggregate growth.

Our results hinge on the assumption that scale effects connected to the size of profits that, in each period, accrue to the incumbent may be negative, positive or null. By focusing on the steady state, we find that, as the degree of scale effects changes, the industrial structure associated to a given level of relative labour endowment may differ significantly. Likewise, there may be also changes in the way the relationship between industrial structure, R&D intensity and long-term aggregate growth is characterised.

If anything, the confrontation with the data suggests that the empirical relevance of our results depends, in general, on the existence of either negative or large positive scale effects. Given this, we underline the pratical importance of distinguishing small but positive from negative scale effects, in particular when theoretical firm size is measured as

[36]Parallel results could be appreciated by conducting a "cross-country" analysis as in Subsection 4.1.1. For instance, if $\bar{\epsilon} < \epsilon < 1$, then the higher $\frac{hH}{lL}$, the higher $\frac{R^*}{Q^*}$ and the lower $\frac{Q^*}{N^*}$, thus implying a negative relationship between firm size and R&D intensity. On the other hand, the higher $\frac{hH}{lL}$, the higher $\frac{X^*}{N^*}$ with an invariant $\frac{R^*}{X^*}$, whatever $\epsilon \neq 1$, implying R&D intensity is independent of firm size.

[37]However, according to the cross-country evidence, relative R&D intensity and relative firm size tend to be negatively correlated, a fact that is compatible with our theoretical results (see fn. 36) if we consider relative firm size measured as $\frac{Q^*}{N^*}$, in the case of small positive scale effects.

technological-knowledge stock per firm. In that case, the predictions of the model suffer a change of sign, which we may term as a "regime switch", with the switch parameter being ϵ.

On the empirical side, further research should be devoted to fill the data gap on the magnitude of scale effects, relative labour endowments measured in efficiency units and dynamic horizontal entry costs, in order to assess the quantitative relevance of our mechanism in explaning the cross-country variability of industrial structure. Some effort might also be devoted to the collection of data on the composition of the technological-knowledge stock between high-tech and low-tech sectors. On the other hand, a larger set of countries than the one used here, in particular including the US and Japan, is desirable in order to guarantee robustness of empirical results.

Finally, in the present paper, the whole of our theoretical analysis was conducted in terms of steady-state equilibrium. The study of the transitional dynamics is, however, an objective for future work. By characterising qualitatively the local dynamics properties in a neighbourhood of the interior steady state, we wish to find to what extent variations in a country's initial conditions (namely the inherited number of firms and stock of technological knowledge) lead to non-monotonic time paths of both industrial structure and skill premium towards the steady state. On the other hand, given the role played by factor endowment and scale effects, it should be only natural to extend our model to an open-economy framework, in particular focusing on intermediate-good international trade and its impact on the cross-country industrial structure.

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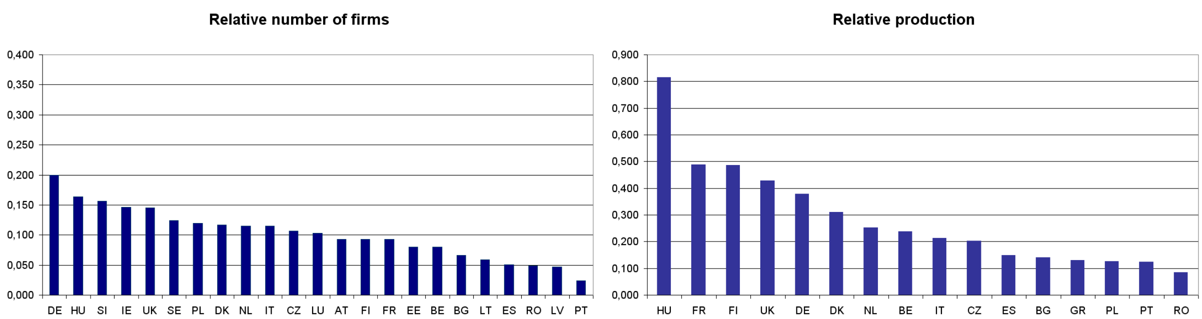
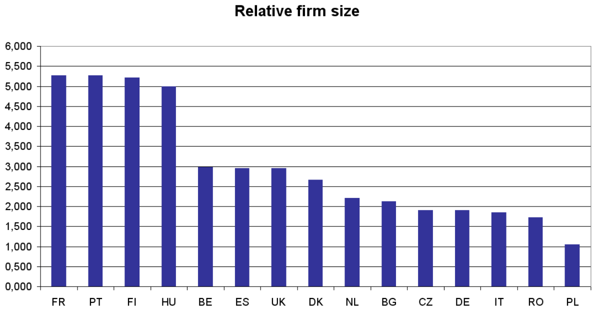
Appendix

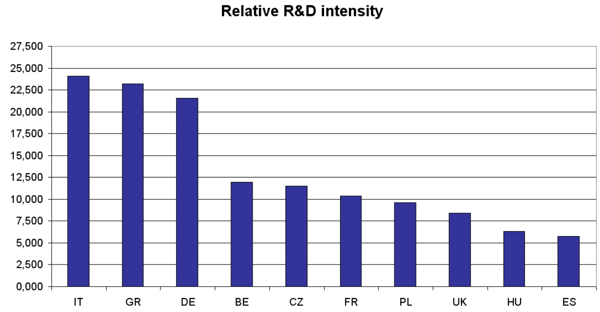
A. Data on high-tech and low-tech sectors

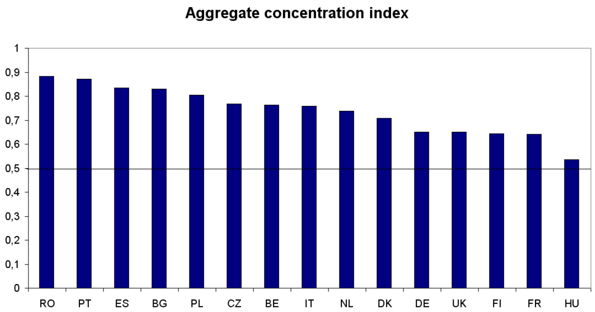
In this appendix we present data with respect to the number of firms, production, average firm size (production/number of firms) and R&D intensity (R&D/production) in high-tech vis-à-vis low-tech sectors, and also aggregate per capita growth rates, all concerning 23 european countries in the period 1995-2005. The source is the Eurostat on-line database, where the OECD classification of high-tech and low-tech sectors is considered. We also compute the aggregate concentration index for each country by applying the data to (77a)-(77b).

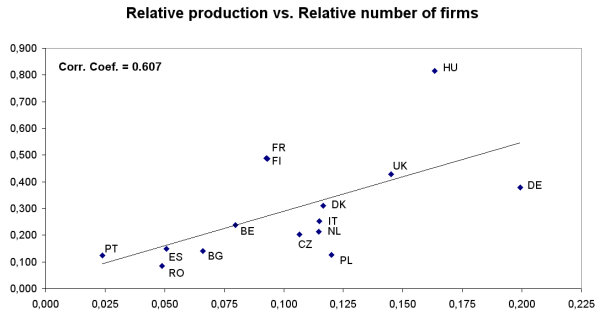
Figure 1

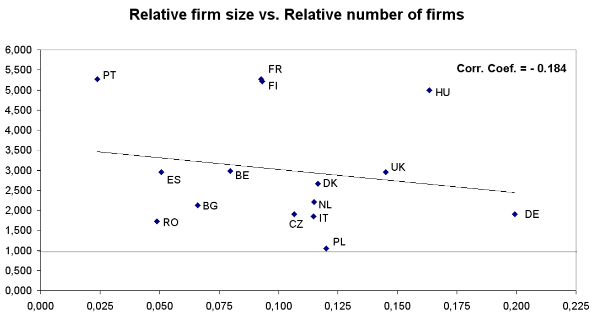


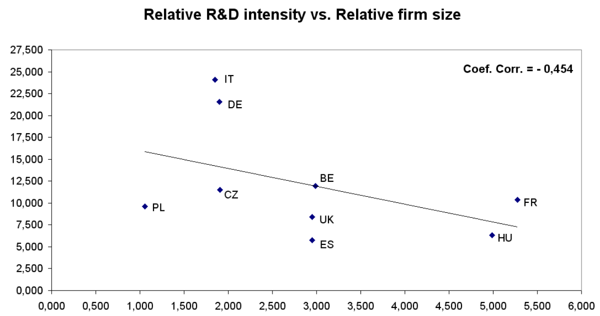

Relative firm size
FR PT FI HU BE ES UK DK NL BG CZ DE IT RO PL

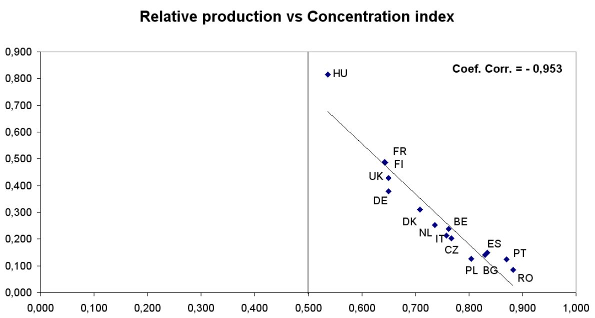

Relative R&D intensity
IT GR DE BE CZ FR PL UK HU ES

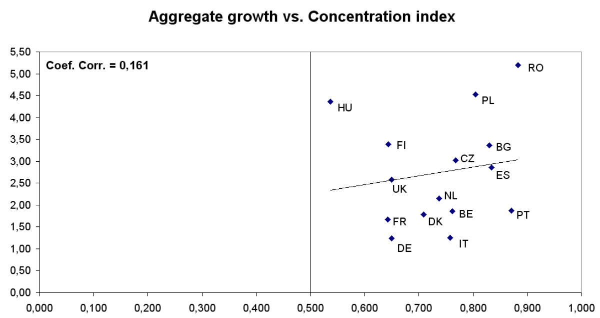

Aggregate concentration index
RO PT ES BG PL CZ BE IT NL DK DE UK FI FR HU


Relative production vs. Relative number of firms
Corr. Coef. = 0.607


Relative firm size vs. Relative number of firms
Corr. Coef. = - 0.184


Relative R&D intensity vs. Relative firm size
Coef. Corr. = - 0,454


Relative production vs Concentration index
Coef. Corr. = - 0,953


Aggregate growth vs. Concentration index
Coef. Corr. = 0,161

B. General equilibrium: derivation of equation (47)

We derive the aggregate resource constraint from the households' balance sheet and flow budget constraint. Firstly, consider the production function (1), such that, given threshold (13), we have firm n value product (time indexes are omitted)

$$P(n)\cdot Y(n)=\begin{cases} P(n)\cdot A\cdot\left[\int_0^{N_L}\left(\lambda^{j_L(\omega)}\cdot x_L(n,\omega)\right)^{1-\alpha}d\omega\right]\left[(1-n)\cdot l\cdot L(n)\right]^{\alpha} & ,\ 0\leq n\leq\bar{n}\\ P(n)\cdot A\cdot\left[\int_0^{N_H}\left(\lambda^{j_H(\omega)}\cdot x_H(n,\omega)\right)^{1-\alpha}d\omega\right]\left[n\cdot h\cdot H(n)\right]^{\alpha} & ,\ \bar{n}\leq n\leq 1\end{cases}\tag{79}$$

Since, in equilibrium, the wage paid to each unit of human capital $m=L,H$ is equal to its marginal value product, we use (79) to get

$$\begin{cases} w_L=\frac{\partial(P(n)Y(n))|_{0\leq n\leq\bar{n}}}{\partial L(n)}\\ w_H=\frac{\partial(P(n)Y(n))|_{\bar{n}\leq n\leq 1}}{\partial H(n)}\end{cases}\Leftrightarrow\begin{cases} w_L\cdot L(n)=\alpha\cdot P(n)\cdot Y(n)\ |_{0\leq n\leq\bar{n}}\\ w_H\cdot H(n)=\alpha\cdot P(n)\cdot Y(n)\ |_{\bar{n}\leq n\leq 1}\end{cases}\tag{80}$$

Aggregating (80) across n and simplifying with (10), yields

$$\begin{cases} \int_0^{\bar{n}}w_L\cdot L(n)dn=\alpha\int_0^{\bar{n}}P(n)\cdot Y(n)dn\\ \int_{\bar{n}}^1 w_H\cdot H(n)dn=\alpha\int_{\bar{n}}^1 P(n)\cdot Y(n)dn\end{cases}\Leftrightarrow\begin{cases} w_L L=\alpha Y_L\\ w_H H=\alpha Y_H\end{cases}\tag{81}$$

where $Y_L=\int_0^{\bar{n}}P(n)Y(n)dn$, $Y_H=\int_{\bar{n}}^1 P(n)Y(n)dn$, such that $Y\equiv Y_L+Y_H=\int_0^1 P(n)Y(n)dn$.

On the other hand, from the derivation of (18), we know that $Y_L=A^{\frac{1}{\alpha}}\cdot(1-\alpha)^{\frac{2(1-\alpha)}{\alpha}}\cdot P_L^{\frac{1}{\alpha}}\cdot l\cdot L\cdot Q_L$ and $Y_H=A^{\frac{1}{\alpha}}\cdot(1-\alpha)^{\frac{2(1-\alpha)}{\alpha}}\cdot P_H^{\frac{1}{\alpha}}\cdot h\cdot H\cdot Q_H$. Also, from the derivation of (17), we learn that $X_L=\int_0^{N_L}X_L(\omega)d\omega=A^{\frac{1}{\alpha}}\cdot(1-\alpha)^{\frac{2}{\alpha}}\cdot P_L^{\frac{1}{\alpha}}\cdot l\cdot L\cdot Q_L$ and $X_H=\int_0^{N_H}X_H(\omega)d\omega=A^{\frac{1}{\alpha}}\cdot(1-\alpha)^{\frac{2}{\alpha}}\cdot P_H^{\frac{1}{\alpha}}\cdot h\cdot H\cdot Q_H$. Therefore, it is easy to show that, given (7),

$$X_m=(1-\alpha)^2\,Y_m\Leftrightarrow p_m X_m=(1-\alpha)\,Y_m\ ,\ m=L,H\tag{82}$$

We put (81) and (82) together to get aggregate gross income

$$Y_m=w_m m+pX_m\ ,\ m=L,H\tag{83}$$

or, since (considering the average intermediate-good sector) total profits in each technology group are $\pi_m N_m=\int_0^{N_m}(p_m(\omega)-1)\cdot X_m(\omega)d\omega=(p-1)X_m$, to get aggregate value added

$$Y_m-X_m=w_m m+\pi_m N_m\ ,\ m=L,H\tag{84}$$

Secondly, consider the households' balance sheet (having in mind the average intermediate-good sector), together with (33),

$$a\equiv a_L+a_H=V_L N_L+V_H N_H=\eta_L N_L+\eta_H N_H\tag{85}$$

which, by time-differentiation, becomes

$$\dot{a} = \eta_L \dot{N}_L + \dot{\eta}_L N_L + \eta_H \dot{N}_H + \dot{\eta}_H N_H \tag{86}$$

Next, we solve (37), in the text, in order to $\dot{\eta}$ and, together with (85) and (42), substitute in (86) to get

$$r\left(a_L + a_H\right) + w_L L + w_H H - C = \eta_L \left(r + I_L\right) N_L - \pi_L N_L + \eta_L \left(\frac{\dot{\pi}_L}{\pi_L} - \frac{1}{\alpha}\frac{\dot{P}_L}{P_L}\right) N_L +$$

$$+\eta_H \left(r + I_H\right) N_H - \pi_H N_H + \eta_H \left(\frac{\dot{\pi}_H}{\pi_H} - \frac{1}{\alpha}\frac{\dot{P}_H}{P_H}\right) N_H + \eta_L \dot{N}_L + \eta_H \dot{N}_H \Leftrightarrow$$

$$\Leftrightarrow w_L L + w_H H + \pi_L N_L + \pi_H N_H = C + I_L \eta_L N_L + I_H \eta_H N_H + \eta_L \dot{N}_L + \eta_H \dot{N}_H +$$

$$+\left(\frac{\dot{\pi}_L}{\pi_L} - \frac{1}{\alpha}\frac{\dot{P}_L}{P_L}\right) \eta_L N_L + \left(\frac{\dot{\pi}_H}{\pi_H} - \frac{1}{\alpha}\frac{\dot{P}_H}{P_H}\right) \eta_H N_H \tag{87}$$

By using (84) in (87) and defining $R_h = R_{hH} + R_{hL}$; $R_v = R_{vH} + R_{vL}$; $R_{hm} = \dot{\eta}_m N_m$ and $R_{vm} = I_m a_m + \left(\frac{\dot{\pi}_m}{\pi_m} - \frac{1}{\alpha}\frac{\dot{P}_m}{P_m}\right) \eta_m N_m$, $m = L, H$, we find

$$Y_L - X_L + Y_H - X_H = C + R_{hH} + R_{hL} + R_{vH} + R_{vL} \Leftrightarrow$$

$$\Leftrightarrow Y = X + C + R_h + R_v$$

which is (47).

Finally, observe that since the real interest rate r consists of dividend payments in units of asset price minus the Poisson death rate, i.e., $r = \frac{\pi_m}{V_m} - I_m$, for each t, then $a_m = V_m N_m \Rightarrow \pi_m N_m = (r + I_m) a_m$.[38] From here and (84), we re-write (42) as

$$\dot{a} = r\left(a_L + a_H\right) + w_L L + w_H H - C =$$

$$= \left(\frac{\pi_L}{V_L} - I_L\right) a_L + \left(\frac{\pi_H}{V_H} - I_H\right) a_H + w_L L + w_H H - C =$$

$$= Y_L - X_L + Y_H - X_H - I_L a_L - I_H a_H - C \tag{88}$$

If we replace (47), solved in order to $R_v + R_h$, in (88), we obtain

$$\dot{a} = R_v + R_h - \left(I_L a_L + I_H a_H\right) \tag{89}$$

[38] See fn. 17, in the text.

which is the accumulation equation for a. The first two terms on the right-hand side of (89) represent the *gross investment* in technological knowledge at time t, whereas the third term represents the *depreciation* (obsolescence) of the existing technological-knowledge stock due to the stochastic arrival of vertical innovations (i.e., as j jumps to $j+1$) in each technological group.

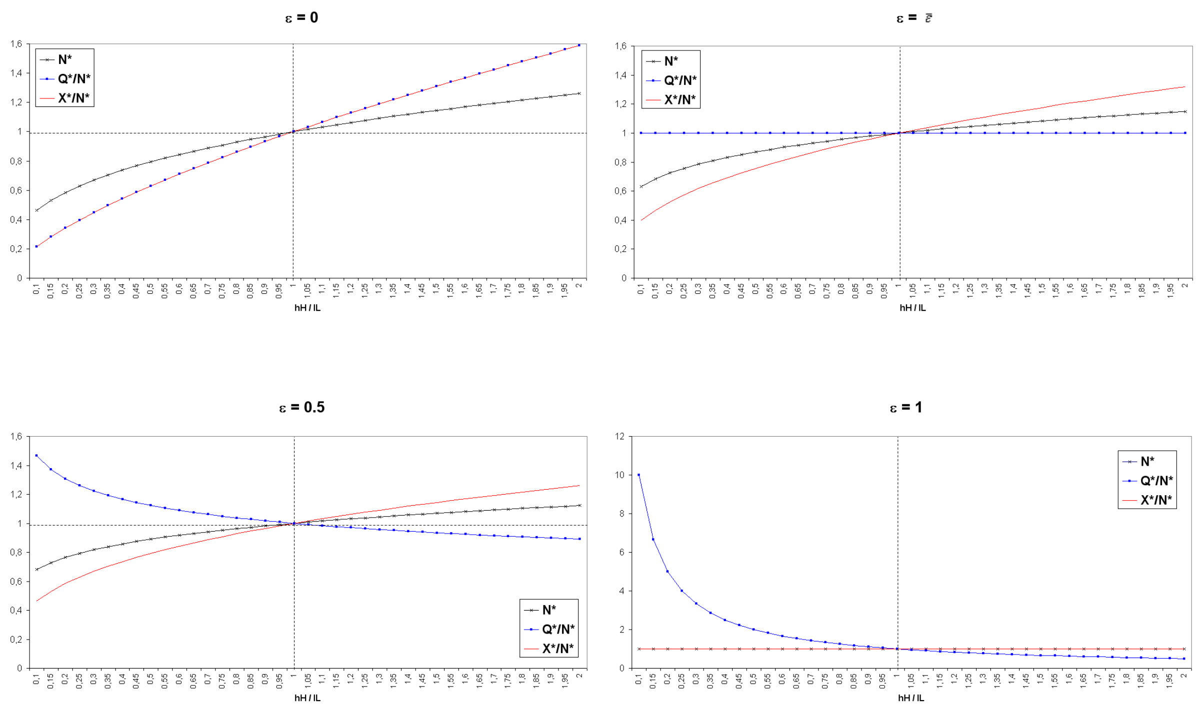
C. Proof of Proposition 1

Here, we give a sketch of the proof of Proposition 1 (the detailed steps can be found in Gil, Brito, and Afonso, 2008): (i), (ii) and (iii) are derived together by time-differentiating the aggregate resource constraint (89), combined with (14), (53); (iv) results from (50a), (51) and (52); finally, (v) results from (i) and (iv).

D. Relative number of firms and relative average firm size

In Figure 2, below, we depict the relationship between relative labour endowment and both the number of firms and (two alternative measures of) average firm size in H-complementary-technology sector vis-à-vis L-complementary-technology sector, for selected degrees of scale-effect removal, ϵ, and given σ and γ.

Figure 2 - Relative number of firms and relative average firm size



E. Concentration index

In this appendix, we present in some detail the two-dimensional aggregate concentration index analysed in the text. Define the "market shares" $u_m \equiv \frac{Q_m}{Q}$, $v_m \equiv \frac{X_m}{X}$, $o_m \equiv \frac{N_m}{N}$ $(m = H, L)$, such that

$$\Sigma_X \equiv v_L \cdot o_L + v_H \cdot o_H = (1 - v_H) \cdot (1 - o_H) + v_H \cdot o_H \tag{90a}$$

$$\Sigma_Q \equiv u_L \cdot o_L + u_H \cdot o_H = (1 - u_H) \cdot (1 - o_H) + u_H \cdot o_H \tag{90b}$$

According to the generic algebraic properties of Σ_Q as defined in (90b), the index has as an upper boundary $\Sigma_Q = 1 \Leftarrow \big(u_m,\, o_m = 0 \Leftrightarrow u_{m'},\, o_{m'} = 1\big)$, i.e., *total concentration of firms and technological-knowledge stock in a single technological sector*; and as a lower boundary $\Sigma_Q = 0 \Leftarrow \big(u_m = 1,\, o_m = 0 \Leftrightarrow u_{m'} = 0,\, o_{m'} = 1\big)$, i.e., *total concentration of firms in one technological sector and of technological-knowledge stock in the other.* Also of interest is $\Sigma_Q = 0.5 \Leftarrow \big(u_m,\, o_m = 0.5 \Leftrightarrow u_{m'},\, o_{m'} = 0.5\big)$, i.e., *uniform distribution of firms and technological-knowledge stock across sectors.* The same should apply to Σ_X in (90a), with the share of technological-knowledge stock replaced by the share of output, $v_m \equiv \frac{X_m}{X}$.

One also obtains $\Sigma_Q = 0.5$ if $u_m = 0.5,\, o_m = 0 \Leftrightarrow u_{m'} = 0.5,\, o_{m'} = 1$, corresponding to a uniform distribution of the technological-knowledge stock but total concentration of firms in one technological sector (or vice-versa). We treat this as a degenerate case in terms of our model, associated with specific levels of scale effects ($\epsilon = \frac{1}{2}$ and $\epsilon = 1$), as shown below (see Proof of Proposition 4.1). This case does not apply to Σ_X.[39]

Yet, given that Σ_Q and Σ_X are also defined as (77b) and (77a) when transposed to our model, we find that, by construction, the upper and lower boundaries are not set alike for the two versions of our concentration measure, as referred in Propositions 4.1 and 4.2, in the text. Next, we give the proof to these propositions.

Proof of Proposition 4.1 ($0 \leq \Sigma_Q \leq 1$): Take (72) and (59) and see that (i) if $0 \leq \epsilon < \frac{1}{2}$ [respectively, $\epsilon > 1$], then $\frac{hH}{lL} > 1 \wedge \frac{hH}{lL} \to \infty \Rightarrow N^*, Q^* \to \infty$ [$\Rightarrow N^*, Q^* \to 0$] and $\frac{hH}{lL} < 1 \wedge \frac{hH}{lL} \to 0 \Rightarrow N^*, Q^* \to 0$ [$\Rightarrow N^*, Q^* \to \infty$], which, substituting in (77a),yields $\Sigma_Q \to 1$ in either case; (ii) if $\frac{1}{2} < \epsilon < 1$, then $\frac{hH}{lL} > 1 \wedge \frac{hH}{lL} \to \infty \Rightarrow N^* \to \infty, Q^* \to 0$ and $\frac{hH}{lL} < 1 \wedge \frac{hH}{lL} \to 0 \Rightarrow N^* \to 0, Q^* \to \infty$, which yields $\Sigma_Q \to 0$ in either case; (iii.a) if $\frac{hH}{lL} = 1, \forall_{\epsilon > 0} \Rightarrow N^*, Q^* = 1$, which now yields $\Sigma_Q = \frac{1}{2}$; (iii.b) if $\epsilon = \frac{1}{2}$ [$\epsilon = 1$], then $\frac{hH}{lL} > 1 \wedge \frac{hH}{lL} \to \infty \Rightarrow N^* \to \infty, Q^* = 1$ [$\Rightarrow N^* = 1, Q^* \to 0$] and $\frac{hH}{lL} < 1 \wedge \frac{hH}{lL} \to 0 \Rightarrow N^* \to 0, Q^* = 1$ [$\Rightarrow N^* = 1, Q^* \to \infty$], which yields $\Sigma_Q \to \frac{1}{2}$. Q.E.D.

Proof of Proposition 4.2 ($\frac{1}{2} \leq \Sigma_X \leq 1$): Take (72) and (74) and see that (i)if $0 \leq \epsilon < 1$ [respectively, $\epsilon > 1$], then $\frac{hH}{lL} > 1 \wedge \frac{hH}{lL} \to \infty \Rightarrow N^*, X^* \to \infty$ [$\Rightarrow N^*, X^* \to 0$]

[39] The generalisation of the concentration index to M technological sectors is straightforward: $\Sigma_M \equiv \overset{M}{\Sigma_i} u_i o_i$, defined in the interval $0 \leq \Sigma_M \leq 1$, with the lower and upper boundaries corresponding to the extreme concentration cases, similarly to the case of $M = 2$, and $\Sigma_M = \frac{1}{M}$ corresponding to the uniform distribution across sectors.

and $\frac{hH}{lL} < 1 \wedge \frac{hH}{lL} \to 0 \Rightarrow N^*, X^* \to 0$ [$\Rightarrow N^*, X^* \to \infty$], which, substituting in (77b),yields $\Sigma_X \to 1$ in either case; and (ii) if $\frac{hH}{lL} = 1, \forall_{\epsilon>0} \vee \epsilon = 1, \forall_{\frac{hH}{lL}>0} \Rightarrow$ $N^*, X^* = 1$, which yields $\Sigma_X = \frac{1}{2}$. Q.E.D.

Figure 3.1, below, depicts the concentration index, defined in terms of the "market shares" $u_H \equiv \frac{Q_H}{Q}$ and $o_H \equiv \frac{N_H}{N}$. Panel (b) represents a correspondence between Σ_Q and u_H, where: $(u_H = 1; \Sigma_Q = 1)$, if $o_H = 1$; $(u_H = 1; \Sigma_Q = 0)$, if $o_H = 0$; $(u_H = 0; \Sigma_Q = 1)$, if $o_H = 1$; and $(u_H = 0; \Sigma_Q = 0)$, if $o_H = 0$. A similar characterisation applies to the correspondence between Σ_Q and o_H in panel (c).

Figure 3.1 - Concentration index and "market shares"

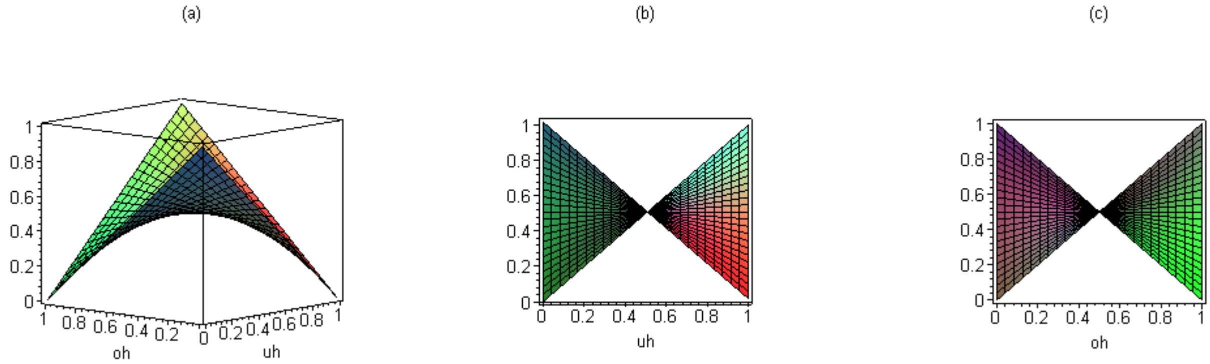


Figure 3.2, below, relates the concentration index with the relative labour endowment. Panel (a) depicts the case of $\epsilon \neq 1$, for Σ_X, and $0 \leq \epsilon < \frac{1}{2}$ and $\epsilon > 1$, for Σ_Q. Panel (b) depicts the case of $\frac{1}{2} < \epsilon < 1$, for Σ_Q. Observe that the concentration index exhibits a symmetric behaviour with respect to $\frac{hH}{lL}$ in geometric terms, i.e., if we define the function $\Sigma \equiv \Sigma\left(\frac{hH}{lL}\right)$, for a given ϵ, then it can be shown that $\Sigma\left(\frac{hH}{lL}\right) = \Sigma\left(\left(\frac{hH}{lL}\right)^{-1}\right)$.[40] On the other hand, it can also be shown that the *higher* σ and γ, the *less pronounced* is the rate of change of Σ with respect to $\frac{hH}{lL}$, through the impact of σ and γ on N^* (see (72), together with (77a) and (77b)).

Figure 3.2 - Concentration index and relative labour endowment

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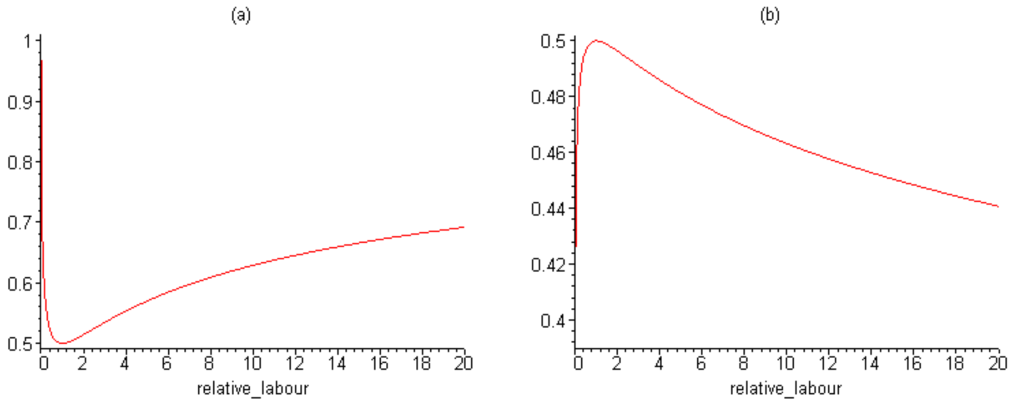
[40]First note, from (59) and (72) in the text, that $\left(\left(\frac{hH}{lL}\right)^{-1}\right)^{1-2\epsilon} = \frac{1}{Q^*}$ and $\left(\left(\frac{hH}{lL}\right)^{-1}\right)^{\frac{1-\epsilon}{\sigma+\gamma+1}} = \frac{1}{N^*}$. Second, let $\Sigma_{\frac{1}{Q}} = \frac{\frac{1}{Q^* \cdot N^*}+1}{\frac{1}{Q^* \cdot N^*}+\frac{1}{Q^*}+\frac{1}{N^*}+1}$. Finally, multiply both numerator and denominator by $Q^* \cdot N^*$, to get $\Sigma_{\frac{1}{Q}} = \frac{1+Q^* \cdot N^*}{1+Q^* \cdot N^* + Q^* + N^*} = \Sigma_Q$ (see (77a)).


(a)
1
0.9
0.8
0.7
0.6
0.5
0 2 4 6 8 10 12 14 16 18 20
relative_labour
(b)
0.5
0.48
0.46
0.44
0.42
0.4
0 2 4 6 8 10 12 14 16 18 20
relative_labour