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Economic Growth in a Two-Agent Economy

Abstract

This paper presents a two-agent economy, in which each agent has a consumption-dependent time preference. The optimal dynamic paths of accumulation will tend to one of many possible steady states, depending on the location of the initial capital level. One of the main results of this model arises in comparison with single-agent models. More precisely, one possible instance of the model consists of a case in which the two agents are such that without interaction one would become "rich" and the other "poor". However, since they share a single production unit, a potential poverty trap may become averted.

1 Introduction.

Since Ramsey's seminal paper, optimal economic growth has been represented by a single agent that maximizes an intertemporal welfare function. His basic construction has remained as the fundamental building block of models of optimal growth. The welfare function consists of the discounted sum of instantaneous utilities. In most models the rates of time preference were assumed to be constant and therefore independent over time. Instead, some researchers considered a variable rate of time preference: Uzawa [24], Beals and Koopmans [3], Iwai [14], Blanchard and Fischer [6] and in the last fifteen years, Mantel [17] [18] [19], Becker and Mulligan [4] and Stern [21]. The key difference among these approaches lies on the particular characteristics of the preference rate. Uzawa, for example, assumes that it increases with the income level, making the optimal growth paths independent of the initial conditions. On the contrary, following a suggestion of Irving Fisher [10], Mantel [19] postulates that the degree of impatience (the rate of time preference) should be a *decreasing* function of consumption and thus, indirectly, of the income level. The lower the income, the higher the sacrifice of postponing present consumption in exchange for future consumption.[1]

[1] A quite different way of including variable discount rates has been largely explored in the literature on Behavioral Economics. The presence of time inconsistency in intertemporal choices indicates that the agents have hyperbolic discount rates (Laibson [15]). On the contrary, in our approach we assume time consistency and focus instead on the existence of multiple steady-states.

A line of work on optimal growth, based on assuming a single agent's decreasing rate of time preference, concludes that these preferences imply a monotone evolution of capital in time, depending on the initial capital levels.[2]. These results seem more satisfactory than Uzawa's. The assumption of decreasing time preference rates, and the conclusion that the initial capital determines the path of growth, are certainly more realistic. In particular, a model with these features can generate poverty traps when the representative agent has a low initial level of capital. However, this is still far away from a realistic depiction of real world economies, which in general do not exhibit monotone paths of growth, and where the heterogeneity among agents cannot be easily reduced to a representation with a single agent. These features of real economies are not unrelated: the *mutual influence* among agents makes capital paths interdependent and therefore more prone to non-monotonicity.

In this paper we present a variation in the literature of optimal growth with varying time preferences. To introduce heterogeneity, our model considers two agents, who differ in their initial capital levels, and therefore in their degrees of impatience, that depend on consumption and indirectly on income. This is intended as a highly stylized representation of closely integrated economies, as for instance those belonging to an economic union. In spite of this empirical interpretation of our framework, we will keep our discussion in an abstract setting. Nevertheless, our final goal is to develop an explicit treatment of interacting economies in growth.

For simplicity we assume that the agents share the yields of a single productive unit (like in a private ownership economy *à la* Arrow-Debreu). This leads to cross-over effects between the individual capital paths. In the induced coordination game, a positive interaction implies that accumulation (deaccumulation) by one agent leads to accumulation (deaccumulation) by the other. This generates monotone paths for both agents. On the other hand, if the interaction is negative, we can obtain non-monotone paths.

In this sense, one of the main results of our model arises in comparison with single-agent models. More precisely, a straightforward implication of the latter is that if two agents choose their optimal plans separately, a kind of poverty trap may arise.[3] That is, two different agents can be such that,

[2] This argument was pursued actively by the late Rolf Mantel [16] [17] [19]. Although Mantel's work along these lines is far less known than his celebrated results on the Sonnenschein-Mantel-Debreu Theorem in General Equilibrium Theory, it follows from the same foundational concerns [23]

[3] A *poverty trap* is defined as any self-reinforcing mechanism which causes poverty to

if there were no interaction, one would become "rich" and the other "poor" because of their respective accumulation policies. In our model, since they interact, an initially growing gap in capital accumulation can be reduced later. In fact, this is the main contribution of this paper: *while in single-agent models low initial levels of capital lead to paths of deaccumulation, and therefore imply the existence of poverty traps, our model allows, in some cases, to avoid them, thanks to the interaction among the agents.*

The model presented here yields the optimal path of accumulation supported by subgame perfect equilibria (the sequences of decisions made in accordance with the solution of Bellman's equation). Nevertheless, the issue of *manipulability* of these outcomes may be raised. We assume here that the rights and allocation of the yields of the shared production unit are common knowledge and furthermore *enforzable* by same agreed external authority. This solves the problem for the moment while it opens venues for further investigation.

The plan of the paper is as follows. Section 2 presents the model. Section 3 is devoted to the characterization of optimal paths. Section 4 characterizes the steady states of the system. Section 5 summarizes the dynamics of this two-agents system. Finally, section 6 discusses our results and compares them with those obtained in an economy with a single agent.

2 The Model

The economy considered here consists of two agents and a single productive unit. We assume, as in a Arrow-Debreu private ownership economy, that both agents get a constant share of the outcome. We want to characterize the equilibrium path of the economy as a function of the preferences, initial levels of capital and outcome shares of the agents.

Each agent i ($i = 1, 2$) has a prospective utility function:

$$W^i({}_0c) = \sum_{s=0}^{\infty} \left\{ \prod_{t=0}^{s-1} \alpha(c_t) \right\} u^i(c_s)$$

where $u^i(c_s)$ is i's instantaneous utility of consuming c_s at s, and the real-valued function $\alpha(c_t)$ is the psychological factor of time preference. In turn,

persist (Azariadis and Stachurski [2]). Surveys of the literature on poverty traps can be found, among others, in Hoff [13], Easterley [9], Azariadis [1], and Azariadis and Stachurski [2].

the rate of time preference function is $\rho(\cdot) \equiv \frac{1}{\alpha(.)} - 1$. Since $\alpha(\cdot)$ is increasing in consumption (and therefore in income), $\alpha(\cdot)$ can also be conceived as being increasing in income. ρ is decreasing, and acts as a discount rate.

The prospective utility function and the psychological time preference have the following properties:

1. $u^i(c)$ and $\alpha(c)$ are continuous on R_+ and twice differentiable for $c > 0$.
2. $u^{i'} > 0 > u^{i''}$, and $\lim_{c\to 0^+} u^{i'}(c) = +\infty$, $\lim_{c\to\infty} u^{i'}(c) = 0$, $u^i(0) \geq 0$.
3. $\alpha^{'} > 0 > \alpha^{''}$, $\alpha(0) > 0$.
4. $0 < \alpha(c) \leq \bar{\alpha} < 1$ for some constant $\bar{\alpha}$, for all $c \geq 0$.
5. $|\alpha^{''} u + \alpha u^{''}| \geq 2|\alpha^{'} u^{'}|$.

It is easy to check that this prospective utility allows us to define an (individual) welfare path $\{W_t^i\}$ such that:

$$W_t^i \equiv W^i[{}_t c] = \sum_{s=t}^{\infty} \left\{ \prod_{v=t}^{s-1} \alpha(c_v) \right\} u^i(c_s)$$

where ${}_t c$ is the consumption path beginning at period t. The sequence $\{W_t^i\}$ satisfies the difference equation:

$$W_t^i = u^i(c_t) + \alpha(c_t) W_{t+1}^i.$$

The second member, $V(c, W^i) \equiv u^i(c) + \alpha(c) W^i$, is called an *utility aggregator*. It has been proved that, with the properties assumed here, this function is continuous, increasing in its arguments and satisfies the Lipschitz condition of order one (Boyd [8]). This shows that successive approximations lead to a single value of the prospective utility.

On the other hand, there exists a single productive unit in the economy. The technology consists in a simple neoclassical aggregate production function, which satisfies Inada's conditions. Therefore, it can be summarized by a real-valued production function $f(k)$, where k is the positive per capita capital. In turn, $f(k)$ has the following properties:

1. It is continuous, twice continuously differentiable for $k > 0$.
2. $f(0) = 0$, $f^{'} > 0$, $f^{''} < 0$, $lim_{k\to 0^+} f^{'}(k) = +\infty$

3. There exists a $k_m > 0$ such that $f(k_m) = k_m$.

4. $0 < k_t < k_m$ for all t.

There is only one good, used both for consumption and for accumulation. Furthermore, $f(\cdot)$ is net of depreciation and of maintenance costs. To simplify the analysis the labor force is assumed constant, and all relevant variables are expressed in *per capita* terms. A capital path ${}_0k$ is admissible and feasible for an initial capital stock k if $k_0 = k$ and for $0 \leq t$:

$$0 \leq k_{t+1} \leq f(k_t)$$

Now we can postulate the optimization problem faced by both agents. Without loss of generality, we assume that the functional form of their psychological factor of time preference $\alpha(\cdot)$ is the same for both agents. Since ρ is decreasing in income, it will reach different values for different initial levels of capital, being this sufficient to induce heterogeneity between the agents, without resorting to different functions of psychological time preference.

Then, for each agent i the problem is to determine the optimal value of the prospective utility, deciding how much to consume and save at each time period, i.e. to find:

$$v^i(k_0^i) = \text{ Max}_{(c_0^i, c_1^i, \ldots)} \sum_{t=0}^{\infty} \beta_t u^i(c_t^i)$$

s.t.

$$k_{t+1}^i \leq \theta_i f(k_t^i + \bar{k}_t^{-i}) - c_t^i$$

$$\beta_{t+1} \leq \alpha(c_t^i)\beta_t$$

$$k_0^i \text{ and } {}_0\bar{k}^{-i}\text{given; } \beta_0 = 1$$

where θ_i is i's share of the income, which is assumed constant and $\theta_i + \theta_{-i} = 1$ (where variables and parameters subindexed by i correspond to agent i while those with subindex $-i$ correspond to the other agent). Moreover, without loss of generality, we assume that $\theta_i > \theta_{-i}$. It is assumed that these shares are enforceable and not open to renegotiation. Of course, c_t^i and k_t^i are, respectively, i's consumption and savings level at period t. The total amount of capital in the economy at t is $k_t = k_t^i + k_t^{-i}$. Finally, notice that β_t actually is a shorthand for the recursively determined weight of $u^i(c_t^i)$, i.e. $\beta_t = \prod_{s=0}^{t-1} \alpha(c_s)$.

Finally, ${}_0\bar{k}^{-i}$ is the optimal plan of the $-i$ agent. That is, each agent's optimal solution constrains the decisions of the other agent. Therefore, we should consider whether there exist equilibria in the choices of ${}_0k^i$ and ${}_0k^{-i}$.

This means that these agents play a coordination game, in which the utilities depend on the accumulation paths determined by the individual choices. More precisely, this is a perfect information sequential game, in which at each step t each agent i has to choose k_t^i. Notice that even if the agents choose their consumptions simultaneously at each period t, this is not an essential deviation from the perfect information setting. Each agent's strategy is $\bar{g}^i(\cdot)$. It assigns to each invested amount (k_t) the value of investment (k_{t+1}^i) that makes optimal the prospective utility of agent i. We have that:

Lemma 1 *The optimal strategy of each agent i, $\bar{g}^{i*}$, is such that $\bar{g}^{i*}(k_t)$=$g^i(k_t^i)$, where g^i is the* policy function *that yields the optimal solution for agent i.*

Proof 1 *It is known that if the instantaneous utility functions are continuous and the spaces of individual consumptions are compact,*[4] *the game has subgame-perfect equilibria [11]. Furthermore, these equilibria obtain by (the limit of) a backward induction process [12]. In this context this means that agent i choses k_{t+1}^i, given $\bar{k}_t^i$ and $\bar{k}_t^{-i}$ (the t-period components of the optimal plans of i and $-i$) satisfying the following equation:*

$$v^i(\bar{k}_t^i) = \ Max_{c_t^i}\left\{u^i(c_t^i) + \alpha(c_t^i)v^i(k_t^i) : c_t^i + k_t^i \leq \theta_i f(k_t^i + \bar{k}_t^{-i})\right\} \qquad (1)$$

By the one-deviation *property (equivalent to subgame-perfection [20]), k_t^i must be the optimal choice of i at period t. But then, since $k_{t+1}^i = \bar{g}^{i*}(k_t^i + \bar{k}_t^{-i})$ and $v^i(k_t^i)$ is the* functional equation *([22]), that is solved by the policy function $k_{t+1}^i = g(k_t^i)$, we have that $\bar{g}(k_t^i + \bar{k}_t^{-i})$=$g(k_t^i)$.*

To simplify matters we focus from now on the policy function, with the understanding that it represents the subgame-perfect equilibrium in the coordination game between i and $-i$. To derive further properties of the optimal solutions for both agents just notice that the optimal solution for (1) must satisfy *Bellman's equation*:

$$v_t^i \ = \ u_t^i + \alpha_t v_{t+1}^i \qquad (2)$$

[4] $c_t \in [0, f(k_m)]$ for all $t \geq 0$.

where $v_t^i = v^i(k_t^i)$, $\alpha_t = \alpha(c_t^i)$ and $v_{t+1}^i = v^i\left(g^i(k_t^i)\right)$.

Notice that, by assumption, u and α are twice continuously differentiable for each t. Therefore, v' and v'' are continuous at each t. Furthermore, the policy function is $g^i(k^i) = \theta_i f(k_t^i + \bar{k}_t^{-i}) - \text{Argmax}_{c_t^i}\left\{u^i(c_t^i) + \alpha(c_t^i)v^i(y) : c_t^i + y \leq \theta_i f(k_t^i + \bar{k}_t^{-i})\right\}$.

The well-known result of Benveniste and Scheinkman ([5]) for discrete dynamic programming problems indicates that if the technology set is convex and with non-empty interior (a condition fulfilled here by the convexity of $f(\cdot)$), and each $\alpha(\cdot)u^i(\cdot)$, when defined over $(k_t, k_{t+1}^i, k_{t+1}^{-i})$ is concave and differentiable, the value function is differentiable in an optimal solution. In fact, αu^i is indeed concave and differentiable, due to the the properties of u, α and f.[5] It follows that v^i, and consequently g^i are continuous and differentiable.

Then, from (2), the necessary first order condition of optimality with respect to the control variable c^i is

$$u_t^{i\prime} + \alpha_t' v_{t+1}^i = \alpha_t v_{t+1}^{i\prime} \tag{3}$$

while the second order condition for a maximum is:

$$u_t^{i\prime\prime} + \alpha_t'' v_{t+1}^i - 2\alpha_t' v_{t+1}^{i\prime} + \alpha_t v_{t+1}^{i\prime\prime} \leq 0 \tag{4}$$

Differentiating (3) for agent i we obtain:

$$\left[u_t^{i\prime\prime} + \alpha_t'' v_{t+1}^i - \alpha_t' v_{t+1}^{i\prime}\right]\theta_i f'\left[dk^i + dk^{-i}\right] = \left[u_t^{i\prime\prime} + \alpha_t'' v_{t+1}^i - 2\alpha_t' v_{t+1}^{i\prime} + \alpha_t v_t^{i\prime\prime}\right] g^{i\prime} dk^i$$

Notice that we have also differentiated the optimal choices of $-i$, treating them as variables and no longer as parameters. This is in order to determine the interaction between the plans of both agents.

Rewriting this last expression we have:

$$g^{i\prime} = \frac{u_t^{i\prime\prime} + \alpha_t'' v_{t+1}^i - \alpha_t' v_{t+1}^{i\prime}}{u_t^{i\prime\prime} + \alpha_t'' v_{t+1}^i - 2\alpha_t' v_{t+1}^{i\prime} + \alpha_t v_t^{i\prime\prime}}\theta_i f' + \frac{u_t^{i\prime\prime} + \alpha_t'' v_{t+1}^i - \alpha_t' v_{t+1}^{i\prime}}{u_t^{i\prime\prime} + \alpha_t'' v_{t+1}^i - 2\alpha_t' v_{t+1}^{i\prime} + \alpha_t v_t^{i\prime\prime}}\theta_i f' \frac{dk^{-i}}{dk^i},$$

and calling

$$G^i = \frac{u_t^{i\prime\prime} + \alpha_t'' v_{t+1}^i - \alpha_t' v_{t+1}^{i\prime}}{u_t^{i\prime\prime} + \alpha_t'' v_{t+1}^i - 2\alpha_t' v_{t+1}^{i\prime} + \alpha_t v_t^{i\prime\prime}}\theta_i f' \tag{5}$$

[5] It is easy to check that $(\alpha u^i)' \geq 0$ and, since $|\alpha'' u + \alpha u''| \geq 2|\alpha' u'|$, $(\alpha u^i)'' \leq 0$.

we have that

$$g^{i\prime} \;=\; G^i \left[1 + \frac{dk^{-i}}{dk^i}\right] \tag{6}$$

where $\frac{dk^{-i}}{dk^i}$ is the coefficient of interaction between the agents' capital paths.

Similarly, for agent $-i$ we have that

$$g^{-i\prime} \;=\; G^{-i} \left[1 + \frac{dk^i}{dk^{-i}}\right] \tag{7}$$

where G^{-i} is analogous to G^i (replacing i for $-i$).

3 Optimal Paths.

Expressions (6) and (7) in the previous section summarize the properties of optimal paths of capital accumulation. Since the policy functions g^i and g^{-i} determine the amount of capital at $t+1$ in terms of the current amount of capital at t, their derivatives define qualitatively the behavior of k^i and k^{-i}.

First, let us note that the sign of g^i defines the monotonicity (or its absence) of $\{k^i_t\}_{t=0}^{\infty}$:

Proposition 1 *If $g^i : K^i \to K^i$, where $K^i \subseteq \Re^+$ is the space of feasible values of k^i, is continuous and differentiable, its first derivative verifies that $g^{i\prime} > 0$ if and only if the capital path $\{k^i_t\}_{t=0}^{\infty}$ is monotone. That is, either $k^i_t \leq k^i_{t+1}$ for all t or $k^i_t \geq k^i_{t+1}$ for all t.*

Proof 1 *If $g^{i\prime}(\cdot) > 0$ it is clear that for $k^i_a \leq k^i_b$, $g^i(k^i_a) \leq g^i(k^i_b)$. Therefore, if $k^i_0 \leq g^i(k^i_0)$ we have by definition that $k^i_0 \leq k^i_1$. This implies that $g^i(k^i_0) \leq g^i(k^i_1)$, or $k^i_1 \leq k^i_2$. By induction we have that, if $k^i_0 \leq g^i(k^i_0)$, $\{k^i_t\}_{t=0}^{\infty}$ is (non-strictly) increasing. On the other hand, if $k^i_0 \geq g^i(k^i_0)$ by a similar argument, it follows that $\{k^i_t\}_{t=0}^{\infty}$ is (non-strictly) decreasing.*
The converse is obvious.

On the other hand, if $g^{i\prime} \leq 0$ we have by the same token that $k^i_0 \leq g^i(k^i_0) = k^i_1 \geq k^i_2 = g^i(k^i_1)$. In other words, $g^{i\prime} \leq 0$ generates a non-monotonic path of capital, *fluctuating* from period to period. Hence, k^i goes up one period and down the next, and so on.

Therefore, to know the behavior of the solutions of (1) for both agents, we have to evaluate the signs of the derivatives of the policy functions. The expressions we have to evaluate are the following:

$$g^{i\prime} = G^i \left[1 + \frac{dk^{-i}}{dk^i}\right]$$

$$g^{-i\prime} = G^{-i} \left[1 + \frac{dk^i}{dk^{-i}}\right]$$

First, let us analyze the sign of each G^i:

$$G^i = \frac{u_t^{i\prime\prime} + \alpha_t^{\prime\prime} v_{t+1}^i - \alpha_t^\prime v_{t+1}^{i\prime}}{u_t^{i\prime\prime} + \alpha_t^{\prime\prime} v_{t+1}^i - 2\alpha_t^\prime v_{t+1}^{i\prime} + \alpha_t v_t^{i\prime\prime}} \theta_i f^\prime$$

where the denominator is, from expression (4), less or equal to 0. According to the properties of the production unit, $\theta_i f^\prime > 0$. The value function verifies $v_t^i \geq 0$ at each t.[6] From equation (3) and the concavity of u^i and α it follows that $v_{t+1}^{i\prime} > 0$. Putting all this together we have that $u_t^{i\prime\prime} + \alpha_t^{\prime\prime} v_{t+1}^i - \alpha_t^\prime v_{t+1}^{i\prime} < 0$. In other words: $G^i > 0$.[7]

Hence,

$$g^{i\prime} > 0 \text{ if and only if } \frac{dk^{-i}}{dk^i} > -1 \tag{8}$$

and

$$g^{-i\prime} > 0 \text{ if and only if } \frac{dk^i}{dk^{-i}} > -1 \tag{9}$$

These equivalences provide the main tools to analyze the behavior of the optimal paths of capital accumulation. First let us note that $\frac{dk^{-i}}{dk^i}$ and $\frac{dk^i}{dk^{-i}}$ at period t may be approximated by $\frac{k_t^{-i} - \bar{k}_t^{-i}}{k_t^i - \bar{k}_t^i}$ and $\frac{k_t^i - \bar{k}_t^i}{k_t^{-i} - \bar{k}_t^{-i}}$, respectively, where $\bar{k}_t^i$ and $\bar{k}_t^{-i}$ are the amounts of capital on the optimal paths at t, while k_t^i and k_t^{-i} are close points that obtain from very small deviations from the optimal path values.

Notice that even if $\frac{dk^{-i}}{dk^i} = \frac{dk^i}{dk^{-i}}$, which reduces the number of free variables required by conditions (8) and (9), we still have three of them for *two* expressions. That is, the system is underdetermined.

[6] It is the solution of the structural problem: a sum of discounted utilities.

[7] The constancy of θ_i and θ_{-i} simplifies matters, but G^i (and G^{-i}) will have the same sign if they are assumed to change in time. For instance, if $\theta_i = \frac{k_t^i}{k_t^i + k_t^{-i}}$, i.e., if the share of the income accrued by i at $t+1$ is the proportion of the total capital invested in t that belongs to i, in the expression of G^i, $\theta_i f^\prime$ is replaced by $\frac{\theta_{-i}}{k_t^i + k_t^{-i}} f + \theta_i f^\prime$ which is also positive.

Therefore, instead of having a single possible optimal path of capital we have a *taxonomy of cases*. To simplify the analysis, we summarize the results as follows:

Proposition 2 *At each t either one of these cases is valid:*

- **Case 1** $sign\left(k_t^i - \bar{k}_t^i\right) = sign\left(k_t^{-i} - \bar{k}_t^{-i}\right)$ *if and only if* $g^{i\prime}, g^{-i\prime} \geq 0$.
- **Case 2** $\left|k_t^i - \bar{k}_t^i\right| > \left|k_t^{-i} - \bar{k}_t^{-i}\right|$ *and* $sign\left(k_t^i - \bar{k}_t^i\right) \neq sign\left(k_t^{-i} - \bar{k}_t^{-i}\right)$ *if and only if* $g^{i\prime} > 0$, $g^{-i\prime} \leq 0$.
- **Case 3** $\left|k_t^i - \bar{k}_t^i\right| \leq \left|k_t^{-i} - \bar{k}_t^{-i}\right|$ *and* $sign\left(k_t^i - \bar{k}_t^i\right) \neq sign\left(k_t^{-i} - \bar{k}_t^{-i}\right)$ *if and only if* $g^{i\prime} \leq 0$ *and* $g^{-i\prime} > 0$.

Proof 2

- **Case 1** $\Rightarrow$) $sign\left(k_t^i - \bar{k}_t^i\right) = sign\left(k_t^{-i} - \bar{k}_t^{-i}\right)$ *implies that* $\frac{dk^{-i}}{dk^i} > 0$. *Then* $\frac{dk^{-i}}{dk^i} > -1$ *and* $\frac{dk^i}{dk^{-i}} > -1$. *That is, according to (8) and (9),* $g^{i\prime}, g^{-i\prime} > 0$.
  $\Leftarrow$) $g^{i\prime}, g^{-i\prime} > 0$ *implies (by (8) and (9)) that* $\frac{dk^{-i}}{dk^i} > -1$ *and* $\frac{dk^i}{dk^{-i}} > -1$. *Suppose that* $sign\left(k_t^i - \bar{k}_t^i\right) \neq sign\left(k_t^{-i} - \bar{k}_t^{-i}\right)$ *and assume, without loss of generality, that* $\left|k_t^i - \bar{k}_t^i\right| > \left|k_t^{-i} - \bar{k}_t^{-i}\right|$, *then,* $\frac{dk^i}{dk^{-i}} < -1$ *while* $\frac{dk^{-i}}{dk^i} > -1$, *absurd. Therefore* $sign\left(k_t^i - \bar{k}_t^i\right) = sign\left(k_t^{-i} - \bar{k}_t^{-i}\right)$
- **Case 2** $\Rightarrow$) $\left|k_t^i - \bar{k}_t^i\right| > \left|k_t^{-i} - \bar{k}_t^{-i}\right|$ *implies that* $\left|\frac{dk^{-i}}{dk^i}\right| < 1$, *while* $sign\left(k_t^i - \bar{k}_t^i\right) \neq sign\left(k_t^{-i} - \bar{k}_t^{-i}\right)$ *implies that* $\frac{dk^{-i}}{dk^i} < 0$. *In other words,* $\frac{dk^{-i}}{dk^i} > -1$, *which in turn means that* $\frac{dk^i}{dk^{-i}} \leq -1$. *That is, according to (8) and (9),* $g^{i\prime} > 0$ *and* $g^{-i\prime} \ngtr 0$.
  $\Leftarrow$) $g^{i\prime} > 0$, $g^{-i\prime} \ngtr 0$ *implies by (8) and (9) that* $\frac{dk^i}{dk^{-i}} > -1$ *and* $\frac{dk^{-i}}{dk^i} \leq -1$. *It is immediate that this implies that* $\left|k_t^i - \bar{k}_t^i\right| > \left|k_t^{-i} - \bar{k}_t^{-i}\right|$ *and* $sign\left(k_t^i - \bar{k}_t^i\right) \neq sign\left(k_t^{-i} - \bar{k}_t^{-i}\right)$.
- **Case 3** *Analogous to the proof of Case 2.*

Case 1 is the simplest: if both agents' optimal capital stocks (the $\bar{k}^i$s) already move in the same direction, even if either one of them is perturbed, their policy functions will still yield monotone paths of accumulation. That is, $\bar{k}^i$ and $\bar{k}^{-i}$ increase or decrease in parallel.

To analyze case 2 we assume without loss of generality that $\bar{k}_1^i - \bar{k}_0^i > 0$ and $\bar{k}_1^{-i} - \bar{k}_0^{-i} < 0$. Then, at $t = 1$ a little perturbation is exerted such that $\left|k_1^i - \bar{k}_1^i\right| > \left|k_1^{-i} - \bar{k}_1^{-i}\right|$ while $k_1^i - \bar{k}_1^i > 0$ and $k_1^{-i} - \bar{k}_1^{-i} < 0$. That is, the perturbation consists in taking some capital away of $-i$ and transferring it to i. Then $g^{i\prime} > 0$ and $g^{-i\prime} \leq 0$. In other words, $\bar{k}^i$ will increase, while $\bar{k}^{-i}$ will change its direction at $t = 1$. That is, after $t = 1$ both capital stocks will move in the same direction, falling into case 1. Notice that once both $g^{i\prime}$ and $g^{-i\prime}$ are non-negative, there is no future period in which they might become negative. Finally, for case 3 the analysis is similar. Hence, either both capital paths are monotone, or one of them increases (decreases) monotonically, while the other reverses its direction after the first period, increasing (decreasing) monotonically thereafter. This means that only if, from period 0 to period 1, one agent accumulates more than the amount that the other deaccumulates, the former will drag the latter towards a path of sustained accumulation.

4 Steady States.

We must complement the previous analysis with a consideration about the long-run growth. In fact, we are interested in the *steady state* solutions to the optimization problems of both agents. That is, we look for capital stocks[8] k^i such that $g^i(k^i) = k^i$. For those values it follows that:

$$v_t^i \;=\; v^i(k_t^i) = \; v^i\left(g^i(k_t^i)\right) \;=\; v_{t+1}^i$$

therefore Bellman's equation (2 becomes:

$$v^i \;=\; u^i + \alpha v^i$$

and the first order condition (3:

$$u^{i\prime} + \alpha' v^i \;=\; \alpha v^{i\prime}$$

Since the control variable reaches a stationary value, Bellman's equation becomes:

$$v^i(k^i) \;=\; u^i(\hat{c}^i) + \alpha(\hat{c}^i) v^i\left(\theta_i f(k^i + k^{-i}) - \hat{c}^i\right) \tag{10}$$

where $\hat{c}^i$ is the steady state value of c^i. This property implies that the accumulation paths of both agents depend of each other:

[8]From now on, in order to simplify the notation, k^i and k^{-i} will denote values in the optimal paths.

Lemma 2 *k^i and k^{-i} must reach their steady states simultaneously.*

Proof 2 *Assume, without loss of generality, that k^i has reached at t its steady state value, $\hat{k}^i$, but k^{-i} still varies. Then, $f(k_t^i + k_t^{-i}) = f(\hat{k}^i + k_t^{-i})$, but since $k_t^{-i} \neq k_{t+1}^{-i}$, $f(\hat{k}^i + k_t^{-i}) \neq f(\hat{k}^i + k_{t+1}^{-i})$. Therefore, according to (10 we have that*

$$\begin{aligned} v^i(\hat{k}^i) &= u^i(\hat{c}^i) + \alpha(\hat{c}^i) v^i \left(\theta_i f(\hat{k}^i + k_t^{-i}) - \hat{c}^i\right) \\ &\neq u^i(\hat{c}^i) + \alpha(\hat{c}^i) v^i \left(\theta_i f(\hat{k}^i + k_{t+1}^{-i}) - \hat{c}^i\right) = v^i(\hat{k}^i) \end{aligned}$$

That is, $v^i(\hat{k}^i) \neq v^i(\hat{k}^i)$. Absurd. Therefore both k^i and k^{-i} must reach their steady states at the same time.

This result should be understood as follows. If the process of capital accumulation of both agents is as prescribed by their policy functions, the convergence to steady states takes infinite periods of time. But if some external shock leaves one of the agents in a steady state, the other must also reach its own. This result is consistent with our analysis of the agents' capital stocks behavior. They move in the same direction (except possibly for the first period) because of their *mutual influence*.

Another condition that follows assuming that the control variable c^i reaches a steady state is the *envelope condition*. Since by definition $g^i(k^i) = \theta_i f(k^i + k^{-i}) - c^i$, differentiating (10) with respect to k^i, we find:

$$v^{i\prime} = \alpha \frac{\partial v^i}{\partial k^i} + \alpha \frac{\partial v^i}{\partial k^{-i}} \frac{dk^{-i}}{dk^i}. \tag{11}$$

Since

$$\frac{\partial v^i}{\partial k^i} = \theta_i v^{i\prime} f' = \frac{\partial v^i}{\partial k^{-i}}$$

(11) becomes

$$1 = \theta_i \alpha f' \left[1 + \frac{dk^{-i}}{dk^i}\right]$$

Since the steady states must be reached simultaneously, the previous condition has to be fulfilled by both of them:

$$1 = \frac{\theta_i}{\theta_{-i}} \frac{\alpha\left(\theta_i f(k^i + k^{-i}) - k^i\right)}{\alpha\left(\theta_{-i} f(k^i + k^{-i}) - k^{-i}\right)} \frac{\left[1 + \frac{dk^{-i}}{dk^i}\right]}{\left[1 + \frac{dk^i}{dk^{-i}}\right]}$$

Introducing a new function to summarize the information involved in this characterization we have

$$
\begin{aligned}
\phi(k^i,k^{-i}) &\equiv \theta_i\alpha\left(\theta_i f(k^i+k^{-i})-g^i(k^i)\right)\left[1+\frac{dk^{-i}}{dk^i}\right]\\
&- \quad \theta_{-i}\alpha\left(\theta_{-i} f(k^i+k^{-i})-g^{-i}(k^{-i})\right)\left[1+\frac{dk^i}{dk^{-i}}\right]
\end{aligned}
$$

The condition $\phi(k^i,k^{-i})=0$ is a necessary and sufficient[9] condition for (k^i,k^{-i}) to be a steady state. Then, the set of steady states of the system is:

$$
\hat{k}^i\times\hat{k}^{-i} \;=\; \left\{(k^i,k^{-i})\in K^i\times K^{-i}:\phi(k^i,k^{-i})\;=0\right\}
$$

The characterization of steady states is given by the following:

Theorem 1 $\emptyset\neq\hat{k}^i\times\hat{k}^{-i} \;=\; \left\{(k^i,k^{-i}):\frac{dk^{-i}}{dk^i}=\frac{\theta_{-i}}{\theta_i}\frac{\alpha\left(\theta_{-i}f(k^i+k^{-i})-k^{-i}\right)}{\alpha(\theta_i f(k^i+k^{-i})-k^i)}\right\}.$

Proof 1 *Consider a generic steady state,* $(\hat{k}^i,\hat{k}^{-i})$*. Then:*

$$
\begin{aligned}
\phi(\hat{k}^i,\hat{k}^{-i}) &= \theta_i\alpha\left(\theta_i f(\hat{k}^i+\hat{k}^{-i})-\hat{k}^i\right)\left[1+\frac{dk^{-i}}{dk^i}\right]\\
&- \quad \theta_{-i}\alpha\left(\theta_{-i} f(\hat{k}^i+\hat{k}^{-i})-\hat{k}^{-i}\right)\left[1+\frac{dk^i}{dk^{-i}}\right] \;=\; 0
\end{aligned}
$$

It follows trivially that

$$
\frac{dk^{-i}}{dk^i} \;=\; \frac{\theta_{-i}}{\theta_i}\frac{\alpha\left(\theta_{-i}f(\hat{k}^i+\hat{k}^{-i})-\hat{k}^{-i}\right)}{\alpha\left(\theta_i f(\hat{k}^i+\hat{k}^{-i})-\hat{k}^i\right)}
$$

On the other hand, for each $(\hat{k}^i,\hat{k}^{-i})$ *such that* $\frac{dk^{-i}}{dk^i} = \frac{\theta_{-i}}{\theta_i}\frac{\alpha\left(\theta_{-i}f(\hat{k}^i+\hat{k}^{-i})-\hat{k}^{-i}\right)}{\alpha\left(\theta_i f(\hat{k}^i+\hat{k}^{-i})-\hat{k}^i\right)}$ *it is trivial that* $\phi\left(\hat{k}^i,\hat{k}^{-i}\right)=0$.

Now let us consider this relation between k^i *and* k^{-i} *at* $(k^i,k^{-i})=(0,0)$*. At that point it is trivial that* $k^{-i}=\frac{\theta_{-i}}{\theta_i}\frac{\alpha(0)}{\alpha(0)}k^i$ *(since at* $(0,0)$*,* $f(0+0)=0$ *and* $g^i(0)=0=g^{-i}(0)$*). Therefore* $\frac{dk^{-i}}{dk^i}=\frac{\theta_{-i}}{\theta_i}\frac{\alpha(0)}{\alpha(0)}$*, i.e.* $(0,0)$ *is a steady state.*

[9]Sufficiency follows from the fact that the envelope condition is defined upon a *solution* to problem (1.

This means that at least $(0, 0)$ is a steady state and that at each steady state the vector field has its direction defined by the relation between the parameters θ and the psychological factors of time preference computed at that point. On the other hand, Theorem 1 is not enough as a characterization of steady states. A steady state like $(0, 0)$ is not economically meaningful. Therefore we are interested in other (positive) values of the steady state variables. We cannot show that these entities exist in general since it depends clearly on the shape of ϕ. We can, instead, provide sufficient conditions for their existence:

Proposition 3 *If there exist two pairs* (k^i_{or}, k^{-i}_{or}) *(close to* $(0, 0)$*) and* $(k^i_m, k^{-i}_m) \gg (0, 0)$ *(with* $k^i_m + k^{-i}_m = k_m$*) such that* $\phi(k^i_m, k^{-i}_m) < 0$ *while* $\phi(k^i_{or}, k^{-i}_{or}) > 0$*, there exist* $(\hat{k}^i, \hat{k}^{-i}) \gg (0, 0)$ *such that* $(\hat{k}^i, \hat{k}^{-i}) \in \hat{k}^i \times \hat{k}^{-i}$*.*

Proof 3 *By assumption, we have that* $(k^i_{or}, k^{-i}_{or}) \in K^- = \{(k^i, k^{-i}) \in K^i \times K^{-i} : \phi(k^i, k^{-i}) \geq 0\}$ *and* $(k^i_m, k^{-i}_m) \in K_- = \{(k^i, k^{-i}) \in K^i \times K^{-i} : \phi(k^i, k^{-i}) \leq 0\}$*. We have that* $K^i \times K^{-i} = K^- \cup K_-$*, and therefore the convex combination of* (k^i_{or}, k^{-i}_{or}) *and* (k^i_m, k^{-i}_m) *is in* $K^- \cup K_-$*. Then, a straightforward application of the Knaster-Kuratowski-Mazurkiewicz Theorem (see Border [7]) yields that there exist* $(\hat{k}^i, \hat{k}^{-i}) \in K^- \cap K_-$*. Furthermore,* $(\hat{k}^i, \hat{k}^{-i})$ *is an element of the segment that joins* (k^i_{or}, k^{-i}_{or}) *and* (k^i_m, k^{-i}_m)*. This means, on one hand, that* $\phi(\hat{k}^i, \hat{k}^{-i}) = 0$*, and on the other that* $(\hat{k}^i, \hat{k}^{-i}) \gg (0, 0)$ *(since only* (k^i_{or}, k^{-i}_{or}) *may have a* 0 *component, but it does not belong to* $K^- \cap K_-$*).*

This means that, if there are at least two pairs of capital levels, one close to the origin, and the other in the technical efficiency frontier, such that the vector field given by g^i and g^{-i} points towards the interior of $K^i \times K^{-i}$ there must exist an interior steady state.

So far, the cardinality of $\hat{k}^i \times \hat{k}^{-i}$ is not determined. But notice that in a compact set, like $K^i \times K^{-i}$, if the steady states are isolated means that they must be finite in number. The advantages of this feature for the analysis of the global dynamics are clear: a finite number of steady states allows a partition of the phase space into a finite collection of attraction basins. The position of the initial capital levels determines the optimal path of the entire system.

As a previous step for this line of analysis, let us give a definition of *non-degeneracy* of steady states:

Definition 1 *A steady state* $(\hat{k}^i, \hat{k}^{-i})$ *is non-degenerate if its gradient* $\nabla\phi(\hat{k}^i, \hat{k}^{-i}) = (\frac{\partial\phi}{\partial k^i}, \frac{\partial\phi}{\partial k^i})$ *verifies that* $\nabla\phi(\hat{k}^i, \hat{k}^{-i}) \cdot (\hat{k}^i, \hat{k}^{-i}) = 0$*, while the matrix of second order derivatives of* ϕ*,* $D^2\phi(\hat{k}^i, \hat{k}^{-i})$ *is either negative or positive definite.*

That is, a steady state is non-degenerate if ϕ attains a local maximum or minimum at the steady state. Therefore:

Lemma 3 *If each steady state* $(\hat{k}^i, \hat{k}^{-i})$ *is non-degenerate, the steady states are isolated and finite in number.*

Proof 3 *Given a non-degenerate steady state* $(\hat{k}^i, \hat{k}^{-i})$ *we have that* $\phi(\hat{k}^i, \hat{k}^{-i}) = 0$*. Consider a linear approximation of* ϕ *at any point* (k^i, k^{-i}) *in a close neighborhood of* $(\hat{k}^i, \hat{k}^{-i})$*:*

$$\phi(k^i, k^{-i}) \;=\; \phi(\hat{k}^i, \hat{k}^{-i}) + \frac{\partial\phi}{\partial k^i}[k^i - \hat{k}^i] + \frac{\partial\phi}{\partial k^{-i}}[k^{-i} - \hat{k}^{-i}] + \frac{1}{2}\left\{\frac{\partial^2\phi}{\partial k^{i2}}[k^i - \hat{k}^i]^2 + \right.$$

$$\left. +\, 2\frac{\partial^2\phi}{\partial k^i \partial k^{-i}}[k^i - \hat{k}^i][k^{-i} - \hat{k}^{-i}] + \frac{\partial^2\phi}{\partial k^{-i2}}[k^{-i} - \hat{k}^{-i}]^2\right\}$$

This expression can be more concisely expressed in the following form:

$$\phi(k^i, k^{-i}) \;= \phi(\hat{k}^i, \hat{k}^{-i}) + \nabla\phi(\hat{k}^i, \hat{k}^{-i}) \cdot \left(k^i - \hat{k}^i, k^{-i} - \hat{k}^{-i}\right) +$$

$$\frac{1}{2}\left\{\left(k^i - \hat{k}^i, k^{-i} - \hat{k}^{-i}\right) \cdot \begin{bmatrix} \frac{\partial^2\phi}{\partial k^{i2}} & \frac{\partial^2\phi}{\partial k^i \partial k^{-i}} \\ \frac{\partial^2\phi}{\partial k^i \partial k^{-i}} & \frac{\partial^2\phi}{\partial k^{-i2}} \end{bmatrix} \cdot \begin{pmatrix} k^i - \hat{k}^i \\ k^{-i} - \hat{k}^{-i} \end{pmatrix}\right\}$$

The first term in the second member is zero because $(\hat{k}^i, \hat{k}^{-i})$ *is a steady state. The second term can be rewritten as:*

$$\nabla\phi(\hat{k}^i, \hat{k}^{-i})\left(k^i - \hat{k}^i, k^{-i} - \hat{k}^{-i}\right) \;=\; \nabla\phi(\hat{k}^i, \hat{k}^{-i})\left(k^i, k^{-i}\right) - \nabla\phi(\hat{k}^i, \hat{k}^{-i})\left(\hat{k}^i, \hat{k}^{-i}\right)$$

where $\nabla\phi(\hat{k}^i, \hat{k}^{-i})\left(\hat{k}^i, \hat{k}^{-i}\right) = 0$ *because of non-degeneracy. On the other hand, since* $D^2\phi(\hat{k}^i, \hat{k}^{-i})$ *is either positive or negative definite we have that in case* $\nabla\phi(\hat{k}^i, \hat{k}^{-i}) \neq 0$ *then either* $\frac{\partial\phi}{\partial k^i}, \frac{\partial\phi}{\partial k^{-i}} \geq 0$ *or* $\frac{\partial\phi}{\partial k^i}, \frac{\partial\phi}{\partial k^{-i}} \leq 0$*. In either case, the sign of the entire expression is either non-negative (when* $D^2\phi(\hat{k}^i, \hat{k}^{-i})$ *is positive definite) or non-positive (when* $D^2\phi(\hat{k}^i, \hat{k}^{-i})$ *is negative definite). Finally, consider the expression inside the curly brackets:*

$$\left(k^i - \hat{k}^i, k^{-i} - \hat{k}^{-i}\right) D^2\phi(\hat{k}^i, \hat{k}^{-i}) \begin{pmatrix} k^i - \hat{k}^i \\ k^{-i} - \hat{k}^{-i} \end{pmatrix}$$

which is strictly negative when $D^2\phi(\hat{k}^i, \hat{k}^{-i})$ is negative definite or strictly positive when it is positive definite. Therefore, in a neighborhood of $(\hat{k}^i, \hat{k}^{-i})$, $\phi \neq 0$, being the steady state isolated. Therefore, the number of steady states must be countable. On the other hand, since $K^i \times K^{-i}$ is closed and bounded, the total number of steady states must be finite.

5 The Structure of the Phase Space.

We can summarize all the previous results in order to characterize the vector field $(g^i(\cdot), g^{-i}(\cdot))$. If there exist non-trivial and non-degenerate steady states, the elements in $\hat{k}^i \times \hat{k}^{-i}$ are finite, and therefore they can be enumerated. Denoting a steady state $(\hat{k}^i, \hat{k}^{-i})$ as $\hat{k}_{i,-i}$, the enumeration of steady states is $\hat{k}^i \times \hat{k}^{-i} = \{\hat{k}^0_{i,-i}, \ldots, \hat{k}^n_{i,-i}\}$, where $\hat{k}^0_{i,-i} = (0,0)$ and $\hat{k}^n_{i,-i}$ lies, eventually, on the boundary $\{(k^i, k^{-i}) : k^i + k^{-i} = k_m\}$. Then, the entire structure of the vector field on $K^i \times K^{-i}$ can be described in terms of the behavior of $(g^i(\cdot), g^{-i}(\cdot))$ related to the elements of $\hat{k}^i \times \hat{k}^{-i}$:

Theorem 2 *For the initial capital level (k^i_0, k^{-i}_0) we have four possible cases:*

- **Case 1** $(k^i_0, k^{-i}_0) = \hat{k}^j_{i,-i}$, *for $j \in \{0, \ldots, n\}$. Then the system remains at (k^i_0, k^{-i}_0).*
- **Case 2** (k^i_0, k^{-i}_0) *is such that (without loss of generality)* $k^i_0 > Max_{\hat{k}^j_{i,-i}} \hat{k}^i$ *while* $k^{-i}_0 < Max_{\hat{k}^j_{i,-i}} \hat{k}^{-i}$. *Then* $sign\left(g^{i\prime}(k^i_0)\right) \neq sign\left(g^{-i\prime}(k^{-i}_0)\right)$.
- **Case 3** (k^i_0, k^{-i}_0) *is such that* $k^i_0 > Max_{\hat{k}^j_{i,-i}} \hat{k}^i$ *and* $k^{-i}_0 > Max_{\hat{k}^j_{i,-i}} \hat{k}^{-i}$. *Then* $g^{i\prime}(k^i_0), g^{-i\prime}(k^{-i}_0) \geq 0$.
- **Case 4** $\hat{k}^0_{i,-i} \leq (k^i_0, k^{-i}_0) \leq \hat{k}^n_{i,-i}$, *then* $g^{i\prime}(k^i_t) \geq 0$ *and* $g^{-i\prime}(k^{-i}_t) \geq 0$, *for* $t \geq 1$.

Proof 2 *The first case is immediate from the definition of steady state. The second case requires to note that since* $k^i_0 > Max_{\hat{k}^j_{i,-i}} \hat{k}^i$ *and* $k^{-i}_0 < Max_{\hat{k}^j_{i,-i}} \hat{k}^i$,$k^i$ *has to decrease towards a steady state, while* k^{-i} *can increase or remain the same. That is, the signs of* $g^{i\prime}(k^i_0)$ *and* $g^{-i\prime}(k^{-i}_0)$ *are different. Case* 3 *is trivial, since both variables must decrease towards a stable steady* *state; thus, the derivative of the policy functions are non-negative. The last case follows considering (by the finiteness of the steady states) that there must exist a pair of steady states* $\hat{k}^j_{i,-i}$ *and* $\hat{k}^l_{i,-i}$ *such that* (k^i_0, k^{-i}_0) *verifies that* $\hat{k}^j_{i,-i} \leq (k^i_0, k^{-i}_0) \leq \hat{k}^l_{i,-i}$ *and that there are no other pair of steady states* $\hat{k}^r_{i,-i}$ *and* $\hat{k}^s_{i,-i}$, $\hat{k}^j_{i,-i} \leq \hat{k}^r_{i,-i} \leq (k^i_0, k^{-i}_0) \leq \hat{k}^s_{i,-i} \leq \hat{k}^l_{i,-i}$. *On the other hand (without loss of generality),* $\hat{k}^j_{i,-i}$ *is a stable steady state while* $\hat{k}^l_{i,-i}$ *is unstable. Otherwise, if both were unstable there would exist a stable steady state between them, violating our assumption that both were the closest to* (k^i_0, k^{-i}_0). *Similarly, if both were stable there would exist an unstable steady state between them, contradicting again our assumption. Then* (k^i, k^{-i}) *will decrease, after an eventual non-monotone jump at* $t = 1$, *towards* $\hat{k}^j_{i,-i}$. *That is,* $g^i, g^{-i} \geq 0$.[10]

According to this result, any dynamical path of capital accumulation will be, from $t = 1$ on, asymptotically convergent towards a non-degenerate steady state. The structure of a possible vector field on $K^i \times K^{-i}$ is depicted in Figure 1. The following result, which is its analytical counterpart, follows as a corollary of Theorem 2:

Lemma 4 *For each pair of steady states,* $\hat{k}^j_{i,-i}$, $\hat{k}^l_{i,-i}$ *we have that*

1. *either* $\hat{k}^j_{i,-i} < \hat{k}^l_{i,-i}$ *or* $\hat{k}^j_{i,-i} > \hat{k}^l_{i,-i}$. *That is, steady states can be linearly ordered.*
2. *if* $\hat{k}^j_{i,-i} < \hat{k}^l_{i,-i}$, *and both are stable steady states, there must exist another steady state* $\hat{k}^s_{i,-i}$, *such that* $\hat{k}^j_{i,-i} < \hat{k}^s_{i,-i} < \hat{k}^l_{i,-i}$ *and for each* $\hat{k}^j_{i,-i} < (k^i_0, k^{-i}_0) < \hat{k}^l_{i,-i}$ *either* $(k^i_t, k^{-i}_t) \overset{t\to\infty}{\to} \hat{k}^j_{i,-i}$ *or* $(k^i_t, k^{-i}_t) \overset{t\to\infty}{\to} \hat{k}^l_{i,-i}$, *or* $(k^i_t, k^{-i}_t) = \hat{k}^s_{i,-i}$ *for all* t.

Proof 4 *Suppose that* $\hat{k}^{i,j} < \hat{k}^{i,l}$ *and* $\hat{k}^{-i,l} < \hat{k}^{-i,j}$ *and no steady state* $\hat{k}^q_{i,-i}$ *is such that* $\hat{k}^{i,j} < \hat{k}^{i,q} < \hat{k}^{i,l}$ *while* $\hat{k}^{-i,j} > \hat{k}^{-i,q} > \hat{k}^{-i,l}$. *One of* $\hat{k}^j_{i,-i}$, $\hat{k}^l_{i,-i}$ *must be stable, otherwise if both were unstable, there would exist two stable steady states* $\hat{k}^r_{i,-i}$ *and* $\hat{k}^s_{i,-i}$ *such that* $\hat{k}^r_{i,-i} < \hat{k}^j_{i,-i}, \hat{k}^l_{i,-i} < \hat{k}^s_{i,-i}$ *and some of the* (k^i, k^{-i}) *such that* $\hat{k}^{i,j} < k^i < \hat{k}^{i,l}$, $\hat{k}^{i,j} < k^{-i} < \hat{k}^{-i,l}$, *would be attracted towards* $\hat{k}^r_{i,-i}$ *while others towards* $\hat{k}^s_{i,-i}$. *In other words, there would exist an unstable steady state* $\hat{k}^q_{i,-i}$ *such that* $\hat{k}^{i,j} < \hat{k}^{i,q} < \hat{k}^{i,l}$ *while*

[10] Notice that this proof recasts Case 2 for every subset of $\hat{k}^i \times \hat{k}^{-i}$.

$\hat{k}^{-i,j} > \hat{k}^{-i,q} > \hat{k}^{-i,l}$. *Contradiction.*
On the other hand, the same contradiction would arise if both steady states were stable. Then only one of them, say $\hat{k}^j_{i,-i}$ *must be stable. But then, if* (k^i_0, k^{-i}_0) *is such that* $\hat{k}^{i,j} < k^i_0 < \hat{k}^{i,l}$ *while* $\hat{k}^{-i,j} < k^{-i}_0 < \hat{k}^{-i,l}$ *a contradiction follows again since* $g^{i\prime} < 0$ *because of the stability of* $\hat{k}^j_{i,-i}$ *while* $g^{i\prime} \geq 0$ *since the conditions correspond to Case* 4 *of Theorem* 2*. Therefore, the steady states are linearly ordered. The validity of the second statement follows immediately from* 1.

Now, to complete the characterization of the dynamics of this system, notice that the basin of attraction towards an interior steady state $\hat{k}^j_{i,-i}$ $(j \neq 0)$ is determined by its two neighboring unstable steady states:

$$\hat{k}^j_{i,-i} \;=\; \{(k^i, k^{-i}) \in K^i \times K^{-i} : \hat{k}^{j-1}_{i,-i} < (k^i, k^{-i}) < \hat{k}^{j+1}_{i,-i}\}$$

while at the boundary, if the steady state is stable we have:

$$\hat{k}^0_{i,-i} \;=\; \{(k^i, k^{-i}) \in K^i \times K^{-i} : \hat{k}^0_{i,-i} < (k^i, k^{-i}) < \hat{k}^1_{i,-i}\}$$

while if it is unstable:

$$\hat{k}^0_{i,-i} \;=\; \{(0,0)\}$$

In any case, it must be clear that any dynamical path beginning at an element of $K^i \times K^{-i}$ will get, at $t = 1$, to the basin of attraction of a steady state even if $\bigcup_{j\in\{0,\ldots,n\}} \hat{k}^j_{i,-i} \neq K^i \times K^{-i}$.

6 A Golden Rule.

We have shown, until now, that the optimal solutions to the problem represented in (1) can exhibit a multiplicity of steady states. This is because the psychological discount factors, and then the rate of time preference depend on the income level. If they were assumed constant, say α^i, α^{-i}, from the envelope condition (11) we would have that at a steady state $(\hat{k}^i, \hat{k}^{-i})$ (using that at a steady state $\frac{dk^{-i}}{dk^i} = \frac{\theta_{-i}}{\theta_i}\frac{\alpha^{-i}}{\alpha^i}$) :

$$f'(\hat{k}^i + \hat{k}^{-i}) \;=\; \frac{1}{\theta_i\alpha^i + \theta_{-i}\alpha^{-i}}.$$

Since f is concave, there is a unique value of $\hat{k}^i + \hat{k}^{-i}$ that verifies this relation. That is, $\hat{k}^i \times \hat{k}^{-i} \subset \left\{(\hat{k}^i, \hat{k}^{-i}) : f'(\hat{k}^i + \hat{k}^{-i}) \;=\; \frac{1}{\theta_i\alpha^i + \theta_{-i}\alpha^{-i}}\right\} \cup \{(0,0)\}$. But, according to Lemma 4 steady states are linearly ordered. Therefore there can exist only one steady state in $\left\{(\hat{k}^i, \hat{k}^{-i}) : f'(\hat{k}^i + \hat{k}^{-i}) = \frac{1}{\theta_i \alpha^i + \theta_{-i} \alpha^{-i}}\right\}$.

If, instead, the rate of time preference is decreasing in income, we can obtain several steady states. This allows us to compare our results with those in Mantel [19]. There, like in our paper, the dynamical path of accumulation of the single agent in the economy is determined both by the initial capital level and the preferences over time. Besides, it follows also that only one steady state exists with a constant or increasing rate of time preference, and several otherwise.

From the results in the previous sections follows that at steady state the marginal productivity of (total) capital equals an expression based on the rates of time preference. This relation is actually a version of the *modified Golden Rule* of Economic Growth theory. More precisely:

Proposition 4 *If* $f'(k^i + k^{-i}) > (<, =) \frac{1}{\theta_i \alpha\left(\theta_i f(\hat{k}^i + \hat{k}^{-i}) - \hat{k}^i\right) + \theta_{-i} \alpha\left(\theta_{-i} f(\hat{k}^i + \hat{k}^{-i}) - \hat{k}^{-i}\right)}$*, in a neighborhood of a stable steady state* $(\hat{k}^i, \hat{k}^{-i})$*,* (k_t^i, k_t^{-i}) *will increase (decrease, remains constant) towards* $(\hat{k}^i, \hat{k}^{-i})$*.*

Proof 4 *From the envelope condition (11) it is immediate that*

$$f'(\hat{k}^i + \hat{k}^{-i}) = \frac{1}{\theta_i \alpha \left(\theta_i f(\hat{k}^i + \hat{k}^{-i}) - \hat{k}^i\right) + \theta_{-i} \alpha \left(\theta_{-i} f(\hat{k}^i + \hat{k}^{-i}) - \hat{k}^{-i}\right)}$$

On the other hand, if (k^i, k^{-i}) *is such that*

$$f'(k^i + k^{-i}) > \frac{1}{\theta_i \alpha \left(\theta_i f(\hat{k}^i + \hat{k}^{-i}) - \hat{k}^i\right) + \theta_{-i} \alpha \left(\theta_{-i} f(\hat{k}^i + \hat{k}^{-i}) - \hat{k}^{-i}\right)}$$

it means that $\hat{k}^i + \hat{k}^{-i} > k^i + k^{-i}$*. One possible case is that* $(k^i, k^{-i}) \leq (\hat{k}^i, \hat{k}^{-i})$ *and therefore* (k^i, k^{-i}) *has to increase towards* $(\hat{k}^i, \hat{k}^{-i})$*. The other possibility is that, say,* $k^i < \hat{k}^i$ *while* $k^{-i} > \hat{k}^{-i}$*. Using their distances towards their steady state value we have that* $|k^i - \hat{k}^i| > |k^{-i} - \hat{k}^{-i}|$*. This, combined with the fact that sign* $\left(k^i - \hat{k}^i\right) \neq$ *sign* $\left(k^{-i} - \hat{k}^{-i}\right)$ *implies that the conditions of Case* 2 *in proposition* 2 *are verified. Therefore* $g^{i\prime} > 0$ *while* $g^{-i\prime} < 0$*. That is,* k^i *will grow towards* $\hat{k}^i$*, and* k^{-i} *will either jump to* $\hat{k}^{-i}$ *and stay thereafter there or will jump below* $\hat{k}^{-i}$ *and then, after* $t = 1$

grow until it reaches that value.
The case of

$$f'(k^i + k^{-i}) \;<\; \frac{1}{\theta_i\alpha\left(\theta_i f(\hat{k}^i + \hat{k}^{-i}) - \hat{k}^i\right) + \theta_{-i}\alpha\left(\theta_{-i} f(\hat{k}^i + \hat{k}^{-i}) - \hat{k}^{-i}\right)}$$

is analogous.

This result is also similar to the characterization given by Mantel [19], in which the movement of capital is determined by the relation between the marginal productivity of capital and the rate of time preference. Thus, the dynamic path of the economy tends to one of several (stable) steady states, depending on the location of the initial level of capital. According to Lemma 4, either these initial values will be situated on the attraction basin of a stable steady state, or else the path of accumulation will jump into a basin at $t = 1$.

7 Discussion.

An important difference between Mantel's and our approach is the implication of presence of more than one agent; in particular, the fact that the dynamics depends on their interaction. While if agents are left on their own one may fall while the other grows sustained, the interaction makes them move in the same direction after $t = 1$. Therefore, while in a single-agent economy with low initial capital the agent fall in a poverty trap, in our two-agent economy the most wealthy may reverse the motion of the poorest and pull him toward a higher income steady state (see Figure 2). In turn, it is also may happen (almost theoretically) that the poorest agent pushes the richest into a lower steady state. In any case both will move monotonically in the same direction after $t = 1$.

How can be our results interpreted in terms of economic development? This question is natural given the original aims of Mantel, who intended his model to provide an explanation of increasing discrepancies between poor and rich countries. In this sense, he provided a formal argument for the origin of poverty traps. In our model, instead, any initial discrepancy can be reduced, or even eliminated, leading to possibility of avoiding poverty traps. As shown in proposition 2, when the accumulation of one agent surpasses the deaccumulation of the other, the latter reverts his path.

The mechanism that yields this reversion has an interpretation in Development Theory: two countries that constitute a common market, with one investing more than any possible disinvestment of the other, can jointly grow through their interaction. The latter country would then begin to invest and then avoid the poverty trap to which it would fall if it were on its own.

On the other hand, as said before, the enforzability of property rights is assumed along the whole exercise. While this allows to dispose of the assignation game between both parties, it leaves for further work the issue of including it in the framework.

Finally, a natural extension of this work is to incorporate specific functional forms and solve the model explicitly, in order to find out the conditions under which poverty traps may be avoided.

8 Figures.

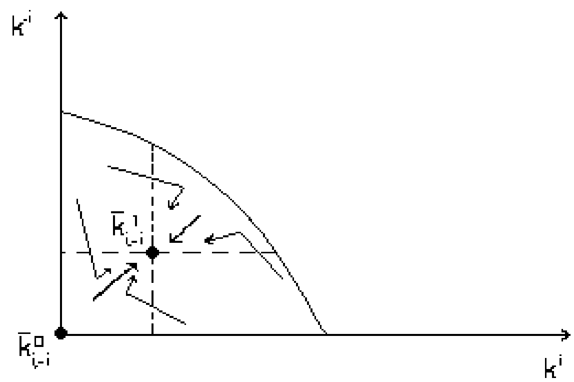

Figure 1
Vector field over the set of feasible states of the economy
(with a single interior stable steady state)

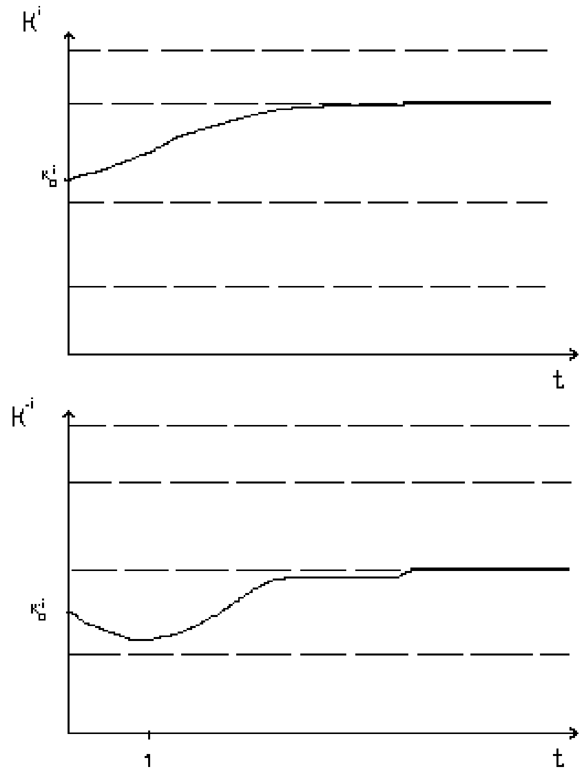

Figure 2
Dynamical paths of capital accumulation with reversion

Note: broken lines indicate non-zero steady states; time paths made continuous for clarity

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