Labour Mobility and Endogenous Growth*

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Preliminary Version

Abstract

Growth models of learning-by-doing assume that the knowledge learned in production gets freely and instantly spread to the whole economy. However, the assumption of instant diffusion of knowledge is unrealistic. Diffusion of knowledge takes time and requires some channel of transmission. In this paper we relax this assumption. We present a model where the free and instant diffusion of knowledge may exist only within sectors, but not across sectors. In contrast, diffusion of knowledge across sectors can only occur through the mobility of labor. We investigate the equilibrium outcome of such economy considering two scenarios: full learning-by-doing and partial learning-by-doing. In the first scenario, the production function is an AK function and, obviously, the equilibrium does not exhibit transition. In this case, the equilibrium path coincides with a BGP along which Gross Domestic Product grows at a constant growth rate. In contrast, when there is partial learning, in an equilibrium path, the production function exhibits decreasing returns to capital. As a consequence, the equilibrium exhibits transition. Moreover, when there are complementarities among the different types of workers, the equilibrium converges to a steady state. However, if there is perfect substitution, the equilibrium may converge to a BGP with full labor mobility. In this case, labor mobility allows to escape from a poverty trap.

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1. Introduction

Learning-by-doing has since Arrow (1962) been recognized as an important determinant of firms productivity growth. The main idea is that by producing a product workers gain experience and become more efficient in the production of such a product. Arrow assumes that the knowledge learned in production gets freely and instantly spread to the whole economy. As a result, he obtains aggregate increasing returns and therefore endogenous economic growth. Certainly, however, the assumption of instant diffusion of knowledge is unrealistic. Diffusion of knowledge takes time and requires some channel of transmission. In this paper we relax this Arrow's assumption. We present a model where the free and instant diffusion of knowledge may exist only within sectors, but not across sectors. In contrast, diffusion of knowledge across sectors can only occur through the mobility of labor. In other words, while sectors may have free and instant access to the knowledge developed within their own sector, they can only learn from other sectors by hiring external workers (learning-by-hiring). We investigate the equilibrium outcome of such economy considering two scenarios: full learning-by-doing within the sector and partial or no learning-by-doing within the sector. We aim at investigating whether learning-by-hiring alone can bring sustained economic growth. To be more precise, we want to check how relaxing the assumption of free and instant diffusion of knowledge would change the results in Arrow (1962).

We present an economy with a final good sector and a continuum of intermediate sectors. The only inputs used in the production of the final good are the intermediate goods. Population consists of a constant amount of infinitely lived individuals and we also assume an inelastic labor supply. They all work in the intermediate sectors. By working in one sector they learn the specific knowledge of that sector without any cost (learning-by-doing), so that, at the beginning of the next period there is a positive amount of workers with the knowledge developed in each sector. We assume that workers have short memory and they only remember what they learned in the last period. We consider a constant population and number of sectors. Moreover, all markets are assumed perfectly competitive.

In each period firms may hire workers from their own sector and poach workers from other sectors. In order to attract external workers, sectors must pay them the same wage as they would earn in their previous sector plus the costs of moving across sectors. We assume that the mobility costs are proportional to the wage level. As mentioned above, each worker has embodied the knowledge of the sector he worked in the last period. Therefore, firms learn the knowledge of other sectors by poaching external workers (learning-by-hiring).

We describe the production function of the intermediate sectors as a Cobb-Douglas function with two inputs: human capital and physical capital. Moreover, the human capital measure is a CES function of all the types of workers hired in that sector weighted by the amount of knowledge they have (as in Vilalta-Bufi, 2008). As in Arrow (1962), the learning of one sector is a function of the investment made the last period in that sector. As a result, the level of knowledge in a sector is going to be the accumulated stock of physical capital in that sector. Nevertheless, firms do not take into account this externality in their decision making.

We solve for the symmetric equilibrium. As stated above, we distinguish between the case of full learning-by-doing and partial or zero learning-by-doing. In the case of full learning-by-doing, the equilibrium production function is an AK function. As a consequence, the equilibrium path does not exhibit transition and coincides with a balanced growth path, as in Arrow (1962). Nevertheless, the equilibrium levels of labor mobility and growth rate vary for different range of parameters. In the case of perfect substitution among types of labor, we obtain three possible equilibria: one with no labor mobility (all workers retained), a second one with full labor mobility (no workers retained) and a third one with an indeterminate level of labor mobility. In fact, the relationship between the learning-by-hiring ability and the mobility costs determines in which of the three equilibria the economy converges to. Interestingly, we show that the more labor mobility in equilibrium the faster the economy grows. In contrast, when workers are imperfectly substitutive there is a unique equilibrium with a determined positive level of labor mobility.

When we analyze the case of partial or zero learning-by-doing results are more diverse. Under imperfect substitution of workers we obtain no long-run economic growth, whereas a balanced growth path may be achieved when there is perfect substitution among the different types of workers. Therefore, in this case, labor mobility allows to achieve sustained economic growth. In fact, in this case, there is a poverty trap. Economies with an initial stock of physical capital below a threshold level converge to a steady state equilibrium with no labor mobility and zero output growth, whereas economies with a initial stock of physical capital above this threshold level grow to a balanced growth path with full labor mobility.

By combining in the model learning-by-doing and learning-by-hiring we obtain several interesting results. First of all, larger mobility of workers brings a higher equilibrium outcome, be it a richer steady state, a balanced growth path or higher economic growth. Moreover, our model emphasizes the role of mobility costs on economic growth. In particular, we observe that a policy aimed at reducing these mobility costs may be effective in liberating the economy from a poverty trap. Finally, we show that for economies that take good advantage of the learning-by-hiring and have low mobility costs, even when the diffusion of knowledge occurs through labor mobility, growth may be a long-run phenomenon.

2. The model

Consider an economy with a final good sector and S intermediate sectors. We assume that the number of intermediate goods sectors is constant. The technology in the final goods sector is defined by the following constant elasticity of substitution (CES) production function:

$$Y = \left(S^{v\mu+\mu-1} \int_0^S y_i^{\mu} di \right)^{\frac{1}{\mu}}, \tag{2.1}$$

where Y is Gross Domestic Product (GDP), y_i is the amount of intermediate goods of sector i used in the production of the final good and $\mu \leq 1$ and v are technological parameters. The elasticity of substitution between two intermediate products is measured by $\frac{1}{1-\mu}$. Obviously, when $\mu < 1$ there are complementarities between the intermediate goods that introduce scale effects into the analysis. As Romer (1990) and many others have shown, this scale effects modify the growth rate. In order to focus on the growth effects of labor mobility, we eliminate these scale effects by assuming that $v = 0$.

We assume that firms operate in the final goods sector in a perfectly competitive market and that they solve the following maximization problem: $\max_{y_i} Y - \int_0^S p_i y_i$ subject to (2.1), where p_i is the price of the intermediate goods in units of the final good. From the first order conditions of this maximization problem, we obtain the demand of intermediate goods

$$p_i = \left(\frac{Y}{y_i}\right)^{1-\mu} S^{v\mu+\mu-1}. \tag{2.2}$$

Infinitely lived workers are employed in the intermediate goods sectors. By working in the sector they learn the specific knowledge of that sector without any cost (learning-by-doing), so that, at the beginning of the next period, there is a positive amount of workers with the knowledge developed in each sector. We assume that workers have short memory and they only remember what they learned in the last period. Alternatively, we can think that the relevant knowledge for production is the newest one. In each period firms may hire workers from their own sector and poach workers from other sectors. Denote by λ_i^j the amount of workers from sector j that are hired in sector i. As already stated above, they have embodied knowledge of sector j. We call them poached workers. Similarly, let η_i be the amount of workers of sector i hired by the same sector i, which have knowledge of sector i. We call them retained workers.

Following Arrow (1962), the knowledge of the sector i is a function of the investment made in the last period in that sector. We then assume that the knowledge of sector i coincides with the average stock of physical capital in that sector, $\bar{k}_i$. Learning is an externality and then it is not internalized by companies when taking their own investment decisions. However, the amount of knowledge accumulated is a determinant of the hiring decisions of the firm. In fact, by hiring workers from other sectors, firms can learn from the investment decisions made in other sectors.

In order to include the possibility of learning by doing and of learning by hiring workers from other sectors, we assume that the production function of sector i is

$$y_i = \left[\left(\eta_i \bar{k}_i^{\xi}\right)^{\sigma} + q\int_0^S \left(\lambda_i^j \bar{k}_j\right)^{\sigma} dj\right]^{\frac{\alpha}{\sigma}} k_i^{1-\alpha},$$

where k_i is the stock of physical capital in sector i, $\xi \in [0,1]$ measures the amount learning by doing, $q > 0$ measures the ability of learning-by-hiring of the sector productivity, $\sigma \leq 1$ and determines the elasticity of substitution between different types of workers, $\frac{1}{1-\sigma}$, and α measures the labor income share. In order to compare with the analysis in the endogenous growth literature, we rewrite this function as follows

$$y_i = \underbrace{\left[\left(\eta_i \bar{k}_i^{\xi-1}\right)^{\sigma} + q\int_0^S \left(\frac{\lambda_i^j \bar{k}_j}{\bar{k}_i}\right)^{\sigma} dj\right]^{\frac{\alpha}{\sigma}}}_{\psi_i} \bar{k}_i^{\alpha} k_i^{1-\alpha}. \tag{2.3}$$

The endogenous growth literature (see Barro, 1990, Rebelo, 1991 and Romer, 1986, among many others) assumes that ψ_i is a technological parameter. In this paper, we endogenize this technological level by analyzing the micro foundations of learning. We show that it also depends on the hiring decisions and, specifically, on the ability to both retain workers and to hire workers from other sectors.

Note that if $\xi = 1$ then there is full learning à la Arrow in our model, that is, sector i fully learns from the investment in sector i. However, with $\xi = 0$, sectors only learn from investments in other sectors. For intermediate values of ξ there is partial learning à la Arrow from the own sector. Independently of ξ, we assume that sectors can always learn from other sectors through labor mobility. In the analysis we will distinguish the case with full learning à la Arrow, $\xi = 1$, and the case of partial or no learning from the own sector, $\xi < 1$. These two cases will lead to different results.

Firms in each sector maximize profits in a perfect competitive market

$$\max_{\eta_i, \lambda_i^j, k_i} p_i y_i - (r+\delta) k_i - \int_0^S w_i^j \lambda_i^j dj - w_i^i \eta_i,$$

subject to (2.3), where r is the rental cost of capital, $\delta \in (0,1)$ is the depreciation rate, w_i^j is the salary paid in sector i to those workers hired in sector j and w_i^i is the salary paid in sector i to those workers hired in the same sector. The firsts order conditions with respect to η_i, λ_i^j, and k_i are, respectively,

$$\alpha P_i \left[\left(\eta_i \bar{k}_i^{\xi-1}\right)^\sigma + q \int_0^S \left(\frac{\lambda_i^j \bar{k}_j}{\bar{k}_i} \right)^\sigma dj \right]^{\frac{\alpha}{\sigma}-1} \eta_i^{\sigma-1} \bar{k}_i^{(\xi-1)\sigma+\alpha} k_i^{1-\alpha} \leq w_i^i, \tag{2.4}$$

$$\alpha P_i \left[\left(\eta_i \bar{k}_i^{\xi-1}\right)^\sigma + q \int_0^S \left(\frac{\lambda_i^j \bar{k}_j}{\bar{k}_i} \right)^\sigma dj \right]^{\frac{\alpha}{\sigma}-1} q(\lambda_i^j)^{\sigma-1} \bar{k}_j^\sigma \bar{k}_i^{\alpha-\sigma} k_i^{1-\alpha} \leq w_i^j, \tag{2.5}$$

$$(1-\alpha) P_i \left[\left(\eta_i \bar{k}_i^{\xi-1}\right)^\sigma + q \int_0^S \left(\frac{\lambda_i^j \bar{k}_j}{\bar{k}_i} \right)^\sigma dj \right]^{\frac{\alpha}{\sigma}} \bar{k}_i^\alpha k_i^{-\alpha} = r + \delta, \tag{2.6}$$

where equations (2.4) and (2.5) hold with equality whenever $\eta_i > 0$ and $\lambda_i^j > 0$.

We assume that there are mobility cost and that these mobility costs are proportional to the wage. Obviously, in order to hire an external worker, the firm has to pay him at least the same wage as in his initial firm plus mobility costs and thus

$$w_i^j \geq m_0 w_i^i$$

where $m_0 - 1 > 0$ measures mobility costs as a percentage of the wage. Perfect competition and free labor mobility implies that the previous relations holds in exact equality, i.e. $w_i^j = m_0 w_i^i$ for all j. Note also that the labor income net of mobility cost obtained by a poached worker is w_i^i. This implies that the net labor income does not depend on the particular sector where workers are employed.

The economy is populated by a large family with N members. The family has a constant number of members and thus population in the economy is also constant. Each member supplies one unit of labor and obtains labor income. *As there is* labor mobility, net labor income equals Nw, where w is the wage obtained by a worker employed in sector i and that already was employed in that sector last period; i.e. $w = w_i^i$ for all i.

This labor income can either be consumed or invested. Then, the budget constraint of the family is

$$Nc + \dot{A} = rA + Nw, \tag{2.7}$$

where c is individual consumption and A is the aggregate stock of financial assets.

The consumers utility function is

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1 - \theta},$$

where $\frac{1}{\theta} > 0$ measures the intertemporal elasticity of substitution. The family maximizes $\int_0^\infty Ne^{-\rho t}u(c)dt$ subject to (2.7). From the first order condition, we obtain that the consumption growth rate satisfies

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho). \tag{2.8}$$

The market clearing conditions in the final goods market implies that

$$Y = C + \dot{K} + \delta K + G, \tag{2.9}$$

where Y is aggregate GDP, $C = Nc$ is the aggregate consumption level, $K = \int_0^S k_i di$ is the aggregate capital stock and G measures the aggregate mobility costs in units of final goods production. These mobility costs are defined as

$$G = \int_0^S \int_0^S (m_0 - 1)w_j^j \lambda_i^j di dj. \tag{2.10}$$

The model is closed with the market clearing condition in the labor market

$$\int_0^S \eta_i di + \int_0^S \int_0^S \lambda_i^j di dj = N. \tag{2.11}$$

3. Equilibrium

In this section we characterize the equilibrium path when we assume that intermediate firms are of identical size. We denote this equilibrium as the symmetric equilibrium path. This assumption and (2.1) imply that GDP is $Y = S^{\frac{v\mu+\mu}{\mu}} y$ and the price level of the intermediate good is $p = S^v$. As mentioned before, in order to prevent any other externality different from learning, we assume that $v = 0$. Then, $p = 1$ and $Y = Sy$.

The symmetric equilibrium assumption and (2.3) imply that the production function in the intermediate sectors simplifies as follows

$$y = \left[(\eta k^{\xi-1})^\sigma + qS\lambda^\sigma\right]^{\frac{\alpha}{\sigma}} k, \tag{3.1}$$

and, thus, GDP is equal to

$$Y = \left[(\eta k^{\xi-1})^\sigma + qS\lambda^\sigma\right]^{\frac{\alpha}{\sigma}} K$$

where $K = sk$ is the aggregate stock of capital. This equation shows the main differences with respect to the Arrow's analysis. In fact, this equation coincides with the GDP equation in Arrow's paper when $q = 0$ and $\xi = 1$. On the one hand, a positive value of q implies that there is learning by hiring in the economy, a possibility that was not present in Arrow's analysis. In fact, in that analysis externalities are due only to learning by working. A main contribution of our paper is to consider the possibility of learning by hiring. This implies that workers are heterogenous. Moreover, as there are different types of workers, complementaries are likely to arise. We have included the possibility of this complemetarities in the model by assuming that σ can be lower than one. As shown in the following section, the introduction of complementarities drives important differences with respect to the previous endogenous growth models. On the other hand, the parameter ξ measures the amount of learning that can be obtained from workers. It is important to differentiate two cases. When $\xi = 1$, we claim that there is full-learning. In this case, the production function is an AK function and, obviously, the equilibrium does not exhibit transition. In contrast, when there is partial learning, $\xi < 1$, the model departures from the standard AK models and it exhibits transition. In the following section, we study the equilibrium dynamics in these two different possible cases and we show that interesting growth patterns arise when there are complemetarities between the different type of workers.

Remark 1. *We implicitly assume that there is always full-learning from hiring workers from other sectors, whereas there is partial learning from workers that are already employed in the same sector. Removing this assumption wouldn't affect the main results of the paper.*

We define the variable $x = \frac{\lambda}{\eta}$ as a measure of labor mobility. Then, in a symmetric equilibrium, the first order conditions, (2.4), (2.5) and (2.6), simplify as follows

$$\alpha \left[(k^{(\xi-1)\sigma} + qSx^{\sigma} \right]^{\frac{\alpha-\sigma}{\sigma}} \eta^{\alpha-1} k^{(\xi-1)\sigma+1} \leq w, \tag{3.2}$$

$$\alpha \left[(k^{(\xi-1)\sigma} + qSx^{\sigma} \right]^{\frac{\alpha-\sigma}{\sigma}} q\eta^{\alpha-1} x^{\sigma-1} k \leq m_0 w, \tag{3.3}$$

$$(1-\alpha) \left[(k^{(\xi-1)\sigma} + qSx^{\sigma} \right]^{\frac{\alpha}{\sigma}} \eta^{\alpha} = r + \delta, \tag{3.4}$$

where equations (3.2) and (3.3) hold with equality whenever $\eta > 0$ and $\lambda > 0$. The production function in the intermediate goods sector simplifies as follows

$$y = \left[(k^{(\xi-1)\sigma} + qSx^{\sigma} \right]^{\frac{\alpha}{\sigma}} \eta^{\alpha} k, \tag{3.5}$$

Combining (2.9) and (2.10), we obtain

$$\dot{k} = y - \frac{N}{S} c - \delta k - (m_0 - 1) w S x \eta, \tag{3.6}$$

and the labor market clearing condition, 2.11, simplifies as follows

$$\eta = \frac{N}{S + S^2 x}. \tag{3.7}$$

Definition 3.1. *An equilibrium of this economy is a path of* $\{x, \eta, c, k, w, r, y\}$ *such that given the initial stock of capital,* k_0*, solves the system of differential equation, (2.8) and (3.6), and satisfies equations (3.2), (3.3), (3.4), (3.5), (3.7).*

In this paper we analyze the path of the symmetric equilibrium. We claim that this path is interior when the amount of retained and poached workers are both positive. In this case, equations (3.2) and (3.3) hold with equality and then we obtain a non-trivial expression of labor mobility

$$x = \left(\frac{qk^{\sigma(1-\xi)}}{m_0} \right)^{\frac{1}{1-\sigma}}. \tag{3.8}$$

However, when this path of the equilibrium is non-interior, then either $x = 0$ or $x \to \infty$. In what follows we study the dynamic equilibrium and we distinguish between the case of full learning, $\xi = 1$, and the case of partial learning, $\xi < 1$.

3.1. Full learning

When $\xi = 1$, (3.8) implies that $x = \left(\frac{q}{m_0}\right)^{\frac{1}{1-\sigma}}$ is constant. In this case, (3.4) implies that the interest rate is constant as in any AK- model. Thus, along the dynamic equilibrium the growth rate is constant, which implies that the equilibrium does not exhibit transition. The equilibrium coincides with a Balanced Growth Path (BGP) along which GDP and consumption grow at the same constant growth rate. However, this growth rate depends on the assumptions on the elasticity of substitution between different workers.

Proposition 3.2. *Assume that* $\xi = 1$ *and* $\sigma < 1$*. Then, consumption, GDP, and capital grow at the following constant growth rate:*

$$\gamma = \frac{1}{\theta} \left[(1-\alpha) \frac{\left(1 + qS\left(\frac{q}{m_0}\right)^{\frac{\sigma}{1-\sigma}}\right)^{\frac{\alpha}{\sigma}} N^{\alpha}}{\left(1 + S\left(\frac{q}{m_0}\right)^{\frac{1}{1-\sigma}}\right)^{\alpha} S^{\alpha}} - \delta - \rho \right].$$

Proof. *In Appendix A we prove that (3.8) holds in equilibrium. Then, by combining (3.8), (3.4) and (2.8) we obtain the growth rate of the BGP.*

Using the growth rate in Proposition 3.2 it can be shown that $\frac{\partial \gamma_c}{\partial m_0} < 0$ and that $\frac{\partial \gamma_c}{\partial q} \gtreqless 0$ when $\sigma \gtreqless 0$. The first derivative implies that mobility costs reduce growth , whereas the second derivative implies that learning-by-hiring affects positively growth as long as the complementarity among different types of workers is not very strong. For elasticities of substitution below one ($\sigma < 0$), learning-by-hiring restrains growth by inducing too much labor mobility.

When we assume perfect substitution among the different types of workers, (3.8) implies that $q = m_0$. Thus, in this case, there is an interior solution. Otherwise, the equilibrium its always in a corner solution. This results are summarized in the following proposition:

Proposition 3.3. *Assume that $\xi = 1$ and $\sigma = 1$. Then,*

a) *If $q < m_0$, there is no labor mobility, $x = 0$ and $\eta = \frac{N}{S}$, and the growth rate equals*

$$\gamma = \frac{1}{\theta}\left[(1-\alpha)\left(\frac{N}{S}\right)^{\alpha} - \delta - \rho\right].$$

b) *If $q > m_0$, there is full labor mobility, $x \to \infty$ and $\eta = 0$, and the growth rate equals:*

$$\gamma = \frac{1}{\theta}\left[(1-\alpha)q^{\alpha}\left(\frac{N}{S}\right)^{\alpha} - \delta - \rho\right].$$

c) *If $q = m_0$, labor mobility is indeterminate and the growth rate equals*

$$\gamma_c = \frac{1}{\theta}\left[(1-\alpha)\left(\frac{N\,(1+qSx)}{S+S^2x}\right)^{\alpha} - \delta - \rho\right].$$

In case a), mobility cost are so large that firms do not poach workers from other sectors. In this case, the growth rate coincides with the growth rate in AK models. In case b), learning by hiring is so large in comparison with mobility costs that no worker is retained in this case. The growth rate is increasing in the parameter q that measures the intensity of learning from hiring. Finally, in case c), firms are indifferent between hiring workers from the same sector or hiring from other sectors. In this case, labor mobility is indeterminate. Thus, in equilibrium $x \in (0, \infty)$ and $\eta \in (0, N)$. Obviously, the growth rate depends on labor mobility. It can be shown that the growth rate increases with x when $q > 1$ and decreases otherwise. Intuitively, the growth rate increases with labor mobility when learning by hiring is more intensive than learning by doing and decreases otherwise.

3.2. Partial learning

When $\xi < 1$, (3.8) implies that labor mobility depends on the stock of capital and it is then not constant. As a consequence, the interest rate is not constant and the equilibrium exhibits transition. However, this transition is driven by the labor market and, in particular, it depends on labor mobility. Again results depend on the value of the elasticity of substitution and on the value of the following parameter

$$\phi = \left(\frac{(\delta+\rho)^{\frac{1}{\alpha}}}{(1-\alpha)^{\frac{1}{\alpha}}N}\right)^{\sigma} S^{2\sigma-1}.$$

Proposition 3.4. *Assume that $\xi < 1$. Then, we distinguish the following cases depending on the value of the elasticity of substitution:*

a) *Assume that $\sigma < 0$. If $q > \phi$ then there is a unique saddle path stable steady state, whereas there is no steady state otherwise.*

b) *Assume that $\sigma \in \left(0, \frac{1}{m_0}\right)$. If $q < \phi$ then there is a unique saddle path stable steady state, if $q = \phi$ then there is either no steady state or a unique steady state and if $q > \phi$ then there is no steady state.*

c) *Assume that* $\sigma \in \left[\frac{1}{m_0}, 1\right)$. *If* $q < \phi$ *then there is a unique steady state, if* $q = \phi$ *then there is either no steady state or a unique steady state and if* $q > \phi$ *then there are either zero, one or two steady states. These steady states are unstable for values of sigma sufficiently far from one.*

d) *Assume that* $\sigma = 1$. *If* $q < \phi$ *then there is a unique saddle path stable steady state with no labor mobility, if* $q \in (\phi, m_0\phi)$ *then there exists an initial value of capital,* $\tilde{k}$, *such that if* $k_0 < \tilde{k}$ *the equilibrium converges to a steady state with no labor mobility and if* $k_0 > \tilde{k}$ *the equilibrium converges to a BGP equilibrium with full labor mobility and sustained growth, and if* $q > m_0\phi$ *then the equilibrium converges to a BGP with full labor mobility and without transition.*

Proof. *See Appendix B for an analysis of the long run equilibrium and Appendix C for an analysis of the stability of these long run equilibria.*

When $\xi < 1$, the production function exhibits decreasing returns to capital. This decreasing returns imply that the equilibrium exhibits transition and converges to a steady state where the variables remain constant. Labor mobility introduces the possibility of multiple steady states. Thus, depending on initial conditions, the economy may converge to a steady states with higher or lower labor mobility.

A particularly interesting case arises when there is perfect substitution among the different types of workers. In this case, sustained growth is possible if the equilibrium converges to the full labor mobility case. Obviously, this depends on the relationship between the intensity of learning by hiring and the mobility costs. In those economies with high mobility cost, the equilibrium converges to a steady state and sustained growth is not possible. In contrast, when the mobility cost are low in comparison to the learning by hiring, then equilibrium converges to a BGP with full labor mobility. For intermediate value of the mobility cost, the economy converges to the steady state if the stock of capital is initially low, whereas converges to the BGP when it is sufficiently large. Thus, in this case, labor mobility cost may imply the convergence to a poverty trap. Those economies that initially are poor converge to a steady state with low labor mobility and zero growth.

4. Concluding remarks

Growth models of learning-by-doing assume that the knowledge learned in production gets freely and instantly spread to the whole economy. This paper relaxes this assumption by presenting a model where the free and instant diffusion of knowledge may exist only within sectors, but not across sectors. As workers can be hired from other sectors, there are different types of workers in the firm. If there is no perfect substitution among these different types of workers, learning will depend on the labor mobility and thus the growth rate of the economy will depend on labor mobility. In this paper, we parametrize these substitution by using a constant elasticity of substitution function. Moreover, we also parametrize the amount of learning that can be obtained from workers. This allows us to differentiate between two cases: full-learning and partial learning. When we assume full-learning, the production function is an AK function and, obviously, the equilibrium does not exhibit transition. In contrast, when there is partial learning, the model departures from the standard AK models and it exhibits transition.

When we assume full-learning, labor mobility is constant and the dynamic equilibrium coincides with a Balanced Growth Path (BGP) along which GDP and consumption grow at the same constant growth rate. However, this growth rate depends on the assumptions on the elasticity of substitution among different types workers. If there are complementarities among these workers, labor mobility is determined and a larger mobility cost reduces labor mobility and the growth rate. In contrast, when there is perfect substitution among the different types of workers, labor mobility is indeterminate and the growth rate depends on the intensity of labor mobility. In fact, the growth rate increases with labor mobility when learning by hiring is more intensive than learning by doing and decreases otherwise.

When we assume partial learning, the production function exhibits decreasing returns to capital. This decreasing returns imply that the equilibrium exhibits transition and converges to a steady state where the variables remain constant. Labor mobility introduces the possibility of multiple steady states. Thus, depending on initial conditions, the economy may converge to a steady states with a higher or a lower labor mobility. A particularly interesting case arises when there is perfect substitution among the different types of workers. In this case, sustained growth is possible if the equilibrium converges to the full labor mobility case. Obviously, this depends on the relationship between the intensity of learning by hiring and the mobility costs. In those economies with high mobility cost, the equilibrium converges to a steady state and sustained growth is not possible. In contrast, when the mobility cost are low in comparison to the intensity of learning by hiring, then the equilibrium converges to a BGP with full labor mobility. For intermediate value of the mobility cost, the economy converges to the steady state if the stock of capital is initially low, whereas converges to the BGP when it is sufficiently large. Thus, in this case, labor mobility introduces the possibility of a poverty trap. Those economies that are initially poor converge to a steady state with low labor mobility and zero growth.

References

[1] Arrow, K.J. (1962). "The Economic Implications of Learning by Doing," *The Review of Economic Studies* 29, 155-173.

[2] Barro, R. (1990). "Government Spending in a Simple Model of Endogenous Growth," *Journal of Political Economy* 98, 103-125.

[3] Rebelo, S. (1991). "Long-Run Policy Analysis and Long-Run Growth," *Journal of Political Economy* 99, 500-521.

[4] Romer, P. (1986). "Increasing Returns and Long Run Growth," *Journal of Political Economy* 94: 1002-1037.

[5] Romer, P. (1990). "Endogenous Technological Change," *Journal of Political Economy*, 98: S71-S102.

[6] Vilalta-Bufí, M. (2008). "On the industry experience premium and labor mobility", Working Paper No. 208, Universitat de Barcelona.

A. A. Full learning and $\sigma < 1$

We have a BGP with a constant x and without transition. Let us check now whether the equilibrium is interior. That is, we want to prove that in equilibrium equations 3.2 and 3.3 hold with equality ($\lambda > 0$, $\eta > 0$) such that equation 3.8 holds:

$$x = \left(\frac{q}{m_0}\right)^{\frac{1}{1-\sigma}}.$$

What would happen if we retain all the workers ($\lambda = x = 0$)? Equation 3.2 is satisfied with equality, while equation 3.3 does not hold with equality. Rewriting equation 3.3 as

$$\alpha\left[(\eta k^{\xi-1})^{\sigma} + qS\lambda^{\sigma}\right]^{\frac{\alpha}{\sigma}-1} q\lambda^{\sigma-1} < \frac{m_0 w}{k}, \tag{A.1}$$

we have that $m_0 w/k$ is finite but the LHS is infinite when $\sigma > 0$. In case that $\sigma < 0$, we have that w/k is positive but the LHS of eq. 3.2 is zero. Thus, this case cannot happen.

What would happen if we poach all the workers ($\lambda = \frac{N}{S^2}$, $\eta = 0$, $x = \infty$)? Equation 3.3 is satisfied with equality, while equation 3.2 does not hold with equality. Rewriting inequation 3.2 as

$$\alpha\left[(\eta k^{\xi-1})^{\sigma} + qS\lambda^{\sigma}\right]^{\frac{\alpha}{\sigma}-1} \eta^{\sigma-1}k^{(\xi-1)\sigma} < \frac{w}{k}, \tag{A.2}$$

we have that w/k is finite but the LHS is infinite when $\sigma > 0$. In case that $\sigma < 0$, we have that w/k is positive but the LHS of eq. 3.2 is infinite, since the squared bracket is equal to one. Thus, this case is not possible either.

So, we proved that the equilibrium is always interior.

B. B. Partial learning. Long run equilibrium

B.1. B.1. Imperfect substitution: $\sigma < 1$

Whenever equation 3.8 holds, we have

$$\gamma_k = \frac{(1-\sigma)}{(1-\xi)\sigma}\gamma_x. \tag{B.1}$$

Combining equations 3.5, 3.7 and 3.8, taking logarithms to the resulting equation and after differentiating with respect to time, we obtain

$$\gamma_y = \left[\frac{-\alpha\,(1+Sx)\,(1-\xi) + [\alpha(1-\xi)\sigma + (1-\sigma)\,(1+Sx)]\,(1+Sm_0x)}{(1+Sm_0x)\,(1+Sx)\,(1-\xi)\sigma}\right]\gamma_x. \tag{B.2}$$

CASE $\sigma > 0$: First we prove that equation 3.8 holds. After, we show that we have a steady state. And, finally, we analyze the existence of the equilibrium.

To prove that equation 3.8 holds, consider first the case where equation 3.2 does not hold with equality. Note that w/k is constant either in a steady state or in a BGP. Since $\eta = 0$, if k is constant then the LHS of equation A.2 is infinite and inequation 3.2 is not satisfied. Thus, if inequation 3.2 is to be satisfied it must occur that $k \to \infty$.

In this case, the LHS of equation A.2 is finite if $\eta^{\sigma-1}k^{(\xi-1)\sigma}$ is finite, too. But using equation A.1 with strict equality we have

$$\eta^{\sigma-1}k^{(\xi-1)\sigma} = \frac{1}{\eta}\left[\left(\frac{m_0 w}{k}\frac{1}{\alpha q\lambda^{\sigma-1}}\right)^{\frac{\sigma}{\alpha-\sigma}} - qS\lambda^{\sigma}\right] \to \infty.$$

Thus, equation 3.2 must hold with equality.

If equation 3.3 does not hold with equality, since $\lambda = 0$ the LHS of equation A.1 is infinite and inequation 3.3 is not satisfied when k is constant. Thus, if inequation 3.3 is to be satisfied it must occur that $k \to \infty$. In this case, since $k \to \infty$ and $x \to 0$, from equation 3.4 we have that $r \to -\delta$, which cannot be. Therefore, equation 3.8 holds.

To prove that we have a steady state, from eq. B.1 we know that k and x grow in the same direction. Next, we prove that γ_y has the same sign of γ_x, so that if there exists a BGP then all the variables grow in the same direction. For y and x to grow in the same direction it must happen that

$$\alpha(1-\xi)\sigma + 1 - \sigma + (1-\sigma)Sx > \alpha(1-\xi)\frac{(1+Sx)}{(1+Sm_0 x)}. \tag{B.3}$$

We prove that equation B.3 is always satisfied. The LHS is linear with intercept at $\alpha(1-\xi)\sigma + 1 - \sigma$ and positive slope. The RHS is decreasing in x and convex. It has intercept $\alpha(1-\xi)$ and a horizontal assymptote at $\alpha(1-\xi)/m_0$ when $x \to 0$. Moreover, at $x = 0$ we have that $\alpha(1-\xi)\sigma + 1 - \sigma > \alpha(1-\xi)$ since $\alpha(1-\xi) < 1$. Thus, equation B.3 is always satisfied.

If a BGP exists, then the growth rates are constant. Let us prove next that this is not the case. In a BGP r is constant, which requires, using equations 3.4, 3.7 and 3.8, the expression

$$\left(\frac{1}{mx^{1-\sigma}} + Sx^{\sigma}\right)^{\frac{\alpha}{\sigma}}(1+Sx)^{-\alpha}$$

to be constant. Taking logarithms and after differentiating, this is true if

$$0 = \left[(1-\sigma) - \frac{Sm}{1/x + Sm} + \frac{\sigma S}{1/x + S}\right]\frac{\dot{x}}{x},$$

i.e., either the squared bracket is zero or $\dot{x}$ is zero and we have a steady state by equation B.1. The squared bracket is zero either for some finite value of x (and then equation B.1 implies that we have a steady state and not a BGP), or for $x = \infty$ (in this case equation B.1 does not hold because equation 3.2 does not bind, but we have proved that equation 3.8 holds). Therefore, we cannot have a BGP. We must have a steady state with some labor mobility.

To prove that we may have zero, one or two steady states, combine equations 2.8, 3.4 and 3.8, to obtain the following equation:

$$\frac{(\delta+\rho)^{\frac{1}{\alpha}}S}{(1-\alpha)^{\frac{1}{\alpha}}N}(1+Sx) = q^{\frac{1}{\sigma}}\left(\frac{1}{m_0 x} + S\right)^{\frac{1}{\sigma}}x. \tag{B.4}$$

The LHS is a straight line with positive slope and intercept $(\delta+\rho)^{\frac{1}{\alpha}}S/(1-\alpha)^{\frac{1}{\alpha}}N$. Since $\sigma > 0$, the RHS is decreasing for $x < (1-\sigma)/\sigma m_0 S$ and increasing otherwise (with a minimum at $x = (1-\sigma)/\sigma m_0 S$). Moreover, the RHS evaluated at $x = 0$ is infinity. Calculating the derivatives we have

$$\frac{\partial RHS}{\partial x} = q^{\frac{1}{\sigma}} \left(\frac{1}{m_0 x} + S \right)^{\frac{1-\sigma}{\sigma}} \left(\frac{\sigma - 1}{\sigma m_0 x} + S \right),$$

$$\frac{\partial^2 RHS}{\partial x^2} = \frac{1-\sigma}{\sigma^2} \left(\frac{1}{m_0 x} + S \right)^{\frac{1-2\sigma}{\sigma}} \frac{q^{\frac{1}{\sigma}}}{m_0^2 x^3} > 0,$$

so that the RHS is convex for all x and all σ. Note that since $\partial^2 RHS/\partial x^2\big|_{x\to\infty} \to 0$, we can have an assymptote.

For a positive σ we can have from zero to two steady states. Let's denote by $\widehat{x}$ the labor mobility level such that the LHS and the RHS of equation B.4 have the same slope. Then, if the $RHS(\widehat{x}) < LHS(\widehat{x})$, there exist two equilibria since the RHS is convex. If the $RHS(\widehat{x}) = LHS(\widehat{x})$, then they are tangent and there exists only one equilibrium. Otherwise, there is no equilibrium. The equation that determines $\widehat{x}$ is

$$q^{\frac{1}{\sigma}} \left(\frac{1}{m_0 \widehat{x}} + S \right)^{\frac{1-\sigma}{\sigma}} \left(\frac{\sigma - 1}{\sigma m_0 \widehat{x}} + S \right) = \frac{(\delta+\rho)^{\frac{1}{\alpha}} S^2}{(1-\alpha)^{\frac{1}{\alpha}} N}. \tag{B.5}$$

In order to check whether $\widehat{x}$ exists, note that the RHS of equation B.5 is a constant and since

$$\frac{\partial LHS}{\partial \hat{x}} = \frac{q^{\frac{1}{\sigma}} \left(\frac{1}{m_0 \widehat{x}} + S \right)^{\frac{1}{\sigma}} (1-\sigma)}{\hat{x} \left(\sigma + m_0 S \hat{x} \sigma \right)^2} > 0,$$

$$\frac{\partial^2 LHS}{\partial \hat{x}^2} = -\frac{q^{\frac{1}{\sigma}} \left(\frac{1}{m_0 \widehat{x}} + S \right)^{\frac{1}{\sigma}} (1-\sigma)(1+\sigma+3m_0 S \sigma \hat{x})}{\hat{x}^2 \left(\sigma + m_0 S \sigma \hat{x} \right)^3} < 0,$$

the LHS is increasing and concave. Moreover, $LHS(0) = -\infty$. The LHS when $x \to \infty$ is $q^{\frac{1}{\sigma}} S^{\frac{1}{\sigma}}$. Therefore, if $q^{\frac{1}{\sigma}} S^{\frac{1}{\sigma}} \geq (\delta+\rho)^{\frac{1}{\alpha}} S^2/(1-\alpha)^{\frac{1}{\alpha}} N$, then $\hat{x}$ exists and is unique. Otherwise, $\hat{x}$ does not exist. When $\hat{x}$ does not exist, then the solution to equation B.4 is unique since the slope of the LHS is always higher than that of the RHS.

Next, we analyze these three cases.

$\sigma>0$ $\sigma<0$

Figure 1. RHS and LHS of equation B.4

a) If $q^{\frac{1}{\sigma}} S^{\frac{1}{\sigma}} > (\delta+\rho)^{\frac{1}{\alpha}} S^2/(1-\alpha)^{\frac{1}{\alpha}} N$, then there will be either zero, one or two steady states. The condition for the existence of the equilibria is equation B.4 evaluated at $\hat{x}$,

$$\frac{(\delta+\rho)^{\frac{1}{\alpha}} S}{(1-\alpha)^{\frac{1}{\alpha}} N} (1 + S\widehat{x}) \geq q^{\frac{1}{\sigma}} \left(\frac{1}{m_0 \widehat{x}} + S \right)^{\frac{1}{\sigma}} \widehat{x}. \tag{B.6}$$

Note that when the RHS and the LHS of equation B.4 are tangent we have one steady state. This happens when the condition B.6 holds with strict equality. In such a case, the steady state is $\hat{x}$.

Substituting from equation B.5 into condition B.6 we obtain

$$(1 + S\widehat{x}) \geq S(1 + Sm_0\widehat{x}) \frac{\sigma\widehat{x}}{(\sigma - 1 + S\sigma m_0\widehat{x})}.$$

Since $\widehat{x} > (1-\sigma)/S\sigma m_0$ (which means that $\widehat{x}$ is in the increasing side of the RHS), then $(\sigma - 1 + S\sigma m_0\widehat{x}) > 0$ and we can simplify the previous expression as

$$\sigma - 1 + S\sigma m_0\widehat{x} - S\widehat{x} \geq 0.$$

Hence, when $\sigma m_0 < 1$ this condition do not longer hold and the equilibrium does not exist. When $\sigma m_0 > 1$ and $\widehat{x} > (1-\sigma)/S(\sigma m_0 - 1)$, then there are two steady states. Moreover, we know that one will be with $x < \widehat{x}$ and the other above this value. When $\sigma m_0 > 1$ and $\widehat{x} < (1-\sigma)/S(\sigma m_0 - 1)$, there is no equilibrium.

b) If $q^{\frac{1}{\sigma}} S^{\frac{1}{\sigma}} = (\delta+\rho)^{\frac{1}{\alpha}} S^2/(1-\alpha)^{\frac{1}{\alpha}} N$, then $\widehat{x} = \infty$, what means that the RHS of equation B.4 has an assymptote with the same solpe than the LHS. Therefore, the condition B.6 holds with strict equality. Then, evaluating equation B.4 at $x = (1-\sigma)/S\sigma m_0$ (the minimum of the RHS), if the RHS is lower than the LHS we have one equilibrium. Otherwise, we have no equilibrium.

c) If $q^{\frac{1}{\sigma}} S^{\frac{1}{\sigma}} < (\delta+\rho)^{\frac{1}{\alpha}} S^2/(1-\alpha)^{\frac{1}{\alpha}} N$, then there will be one equilibrium (it is easy to check that the RHS of equation B.4 is below the LHS as $x \to \infty$).

CASE $\sigma < 0$: First we prove that equation 3.8 holds. After, we show that we have a steady state. And, finally, we analyze the existence of the equilibrium.

To prove that equation 3.8 holds, consider first the case where equation 3.2 does not hold with equality. Note that w/k is constant either in a steady state or in a BGP. Since $\eta = 0$, if k is constant then from equations A.2 and A.1 with strict equality we have

$$\frac{1}{m_0} > \frac{\eta^{\sigma-1} k^{(\xi-1)\sigma}}{q\lambda^{\sigma-1}} \to \infty, \tag{B.7}$$

so that equation A.2 is not satisfied. Thus, if inequation 3.2 is to be satisfied it must occur that $k \to \infty$. In this case, the LHS of equation A.2 is finite if $\eta^{\sigma-1}k^{(\xi-1)\sigma}$ is finite, too. But since equation A.1 with strict equality implies that $\eta^{\sigma}k^{(\xi-1)\sigma}$ is constant, the condition B.7 also applies, so that equation A.2 is not satisfied. Thus, equation 3.2 must hold with equality.

If equation 3.3 does not hold with equality, since $\lambda = 0$ the LHS of equation A.1 is infinite and inequation 3.3 is not satisfied when k is constant. When $k \to \infty$ and $x \to 0$, from equation 3.4 we have that $r \to \infty$, which cannot be. Therefore, equation 3.8 holds.

To prove that we have a steady state, in a BGP exists we have that, as r is constant and $y = (r+\delta)k/(1-\alpha)$, $sign(\gamma_y) = sign(\gamma_k)$. Since from equation B.1 $sign(\gamma_k) = -sign(\gamma_x)$, if a BGP exists, we need y and x to grow in opposite directions. Therefore, from equation B.2 to have a BGP we need the following condition to be satisfied:

$$[\alpha(1-\xi)\sigma + 1 - \sigma] + (1-\sigma)Sx < \alpha(1-\xi)\frac{(1+Sx)}{(1+Sm_0x)}. \tag{B.8}$$

The LHS is linear and increasing with intercept $\alpha(1-\xi)\sigma+1-\sigma>0$. The RHS, as proved in equation B.3, is decreasing and convex, with intercept $\alpha(1-\xi)$. Since $\alpha(1-\xi)\sigma+1-\sigma>\alpha(1-\xi)$, then equation B.8 never holds. Therefore, we have again a steady state.

Next, we prove that this SS exists and is unique. For that we need to study equation B.4 for the case of $\sigma<0$. The LHS of equation B.4 is a straight line with positive slope and intercept $(\delta+\rho)^{\frac{1}{\alpha}}S/(1-\alpha)^{\frac{1}{\alpha}}N$. The RHS is always increasing and is zero when $x=0$. We showed that the RHS is convex for all x and all σ. If $\partial RHS/\partial x|_{x\to\infty}=q^{\frac{1}{\sigma}}S^{\frac{1}{\sigma}}>\partial LHS/\partial x=(\delta+\rho)^{\frac{1}{\alpha}}S^2/(1-\alpha)^{\frac{1}{\alpha}}N$, then they cross only once and the steady state is unique. Otherwise, there is no equilibrium.

B.2. B.2. Perfect substitution: $\sigma=1$

In this case, notice that equations 3.2 and 3.3 cannot be simultaneously holding with equality unless $qk^{1-\xi}=m_0$. Therefore, we should consider three subcases:

1) When $qk^{1-\xi}=m_0$ we have an interior equilibrium, so that firms hire both retained and poached workers and we have a steady state with $k_{SS}=(m_0/q)^{1/(1-\xi)}$. Using equations 2.8, 3.4 and 3.7, and equalizing the growth rate to zero, we obtain

$$x=\frac{(\delta+\rho)^{\frac{1}{\alpha}}m_0S-qN(1-\alpha)^{\frac{1}{\alpha}}}{\left[qN(1-\alpha)^{\frac{1}{\alpha}}-(\delta+\rho)^{\frac{1}{\alpha}}S\right]m_0S}.$$

We need to check that $x>0$. Notice that we will never have the numerator and denominator negative at the same time. Thus, the only way to have a positive x is to have the numerator and denominator positive. This happens if

$$\left(\frac{\delta+\rho}{1-\alpha}\right)^{\frac{1}{\alpha}}\frac{m_0}{q}>\frac{N}{S}\geq\left(\frac{\delta+\rho}{1-\alpha}\right)^{\frac{1}{\alpha}}\frac{1}{q}.$$

Note that $[(\delta+\rho)/(1-\alpha)]^{1/\alpha}(m_0/q)=N/S$ when $x=0$ and $N/S=[(\delta+\rho)/(1-\alpha)]^{1/\alpha}(1/q)$ when $x=\infty$.

2) When $qk^{1-\xi}<m_0$ we have a corner equilibrium without labor mobility, $\lambda=x=0$ and $\eta=N/S$. The growth rate becomes

$$\gamma_c=\frac{1}{\theta}\left[(1-\alpha)\left(\frac{N}{S}\right)^{\alpha}k^{\alpha(\xi-1)}-\delta-\rho\right].$$

If a BGP exists, $k\to\infty$ and then $\gamma_c\to-(\delta+\rho)/\theta$, which cannot be. Therefore, we have a steady state with

$$k=\left(\frac{1-\alpha}{\delta+\rho}\right)^{\frac{1}{\alpha(1-\xi)}}\left(\frac{N}{S}\right)^{\frac{1}{(1-\xi)}}.$$

Recall that we need that $k<(m_0/q)^{1/(1-\xi)}$. Therefore, this equilibrium exists if

$$\frac{N}{S}<\left(\frac{m_0}{q}\right)\left(\frac{\delta+\rho}{1-\alpha}\right)^{\frac{1}{\alpha}}.$$

3) When $qk^{1-\xi} > m_0$ we have a corner equilibrium with full labor mobility, $\eta = 0$ and $\lambda = N/S^2$. Firms will have incentives to hire only poached workers. In this case, from equations 2.8 and 3.4 we get a BGP,

$$\gamma_c = \frac{1}{\theta}\left[(1-\alpha)q^{\alpha}\left(\frac{N}{S}\right)^{\alpha} - \delta - \rho\right].$$

Moreover, there is no transition. We have $\gamma_c \geq 0$ if

$$\frac{N}{S} \geq \left(\frac{\delta+\rho}{1-\alpha}\right)^{\frac{1}{\alpha}}\frac{1}{q}.$$

Note that $\gamma_c = 0$ if $N/S = \left[(\delta+\rho)/(1-\alpha)\right]^{1/\alpha}(1/q)$.

Based on the previous analysis, we can conclude the following:

i) If $N/S < \left[(\delta+\rho)/(1-\alpha)\right]^{1/\alpha}(1/q)$, for any initial level of k we will end up in a steady state with $x = 0$, since even with an initial level of capital k_0 such that $qk_0^{1-\xi} > m_0$ (subcase 3), we have $\gamma_c < 0$, so that k decreases until we have $qk^{1-\xi} \leq m_0$ (subcase 2).

ii) If $N/S > \left[(\delta+\rho)/(1-\alpha)\right]^{1/\alpha}(m_0/q)$ the only possible equilibrium is a BGP with $x = \infty$.

iii) If $\left[(\delta+\rho)/(1-\alpha)\right]^{1/\alpha}(m_0/q) > N/S > \left[(\delta+\rho)/(1-\alpha)\right]^{1/\alpha}(1/q)$, then when $k_0 < (m_0/q)^{1/(1-\xi)}$ we go to the steady state with $x = 0$ and when $k_0 > (m_0/q)^{1/(1-\xi)}$ we are in the case of a BGP with $x = \infty$. Notice that we have a poverty trap in this case. Moreover, if $k_0 = \left(\frac{m_0}{q}\right)^{\frac{1}{1-\xi}}$ we are in an unstable steady state.

C. C. Partial learning. Stability analysis

C.1. C.1. Imperfect substitution: $\sigma < 1$

Recall that we have to distinguish between $\sigma > 0$ and $\sigma < 0$ and that we have steady state(s) if existing. From equations 2.8, 3.4 and 3.7, and 3.2, 3.5, 3.6, 3.7 and 3.8 we have

$$\begin{aligned}
\dot{c} &= \frac{c}{\theta}\left[(1-\alpha)\frac{\left(k^{(\xi-1)\sigma}+qSx^{\sigma}\right)^{\frac{\alpha}{\sigma}}}{(1+Sx)^{\alpha}}\left(\frac{N}{S}\right)^{\alpha} - \delta - \rho\right], \\
\dot{k} &= \frac{k\left(k^{(\xi-1)\sigma}+qSx^{\sigma}\right)^{\frac{\alpha}{\sigma}}}{(1+Sx)^{\alpha}}\left(\frac{N}{S}\right)^{\alpha}\left\{1 - \frac{\alpha(m_0-1)Sxk^{(\xi-1)\sigma}}{\left(k^{(\xi-1)\sigma}+qSx^{\sigma}\right)}\right\} - \frac{N}{S}c - \delta k.
\end{aligned}$$

that, subtituting for x from equation 3.8, become a 2 dynamic equation system in k and c. Linearizing the system around the steady state, we have

$$\begin{bmatrix} \dot{c} \\ \dot{k} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \dot{c}}{\partial c} & \frac{\partial \dot{c}}{\partial k} \\ \frac{\partial \dot{k}}{\partial c} & \frac{\partial \dot{k}}{\partial k} \end{bmatrix}}_{A}\begin{bmatrix} (c-c_{ss}) \\ (k-k_{ss}) \end{bmatrix}.$$

The derivatives evaluated at the steady state[1] are $\partial \dot{c}/\partial c = 0$, $\partial \dot{k}/\partial c = -N/S$,

$$\frac{\partial \dot{c}}{\partial k} = \frac{c}{\theta}\frac{\alpha q(1-\xi)(1-\alpha)\left(\frac{N}{S}\right)^{\alpha}}{(1+Sx)^{1+\alpha}km_0}\left(k^{(\xi-1)\sigma}+qSx^{\sigma}\right)^{\frac{\alpha-\sigma}{\sigma}}\left[\left(\frac{\sigma m_0-1}{1-\sigma}\right)Sx-1\right]x^{\sigma-1},$$

$$\frac{\partial \dot{k}}{\partial k} = \rho+\left(\frac{N}{S}\right)^{\alpha}\frac{\left(k^{(\xi-1)\sigma}+qSx^{\sigma}\right)^{\frac{\alpha}{\sigma}}\alpha\left[x^{-2}\xi\left(1-\sigma\right)+x^{-1}S\left(1-\sigma\right)u_1+xS^3m_0\left(1-\sigma\right)+S^2u_2\right]}{(1+Sx)^{1+\alpha}\left(1-\sigma\right)\left(x^{-1}+m_0S\right)^2},$$

where $u_1 = m_0\left[\xi+\alpha\left(1-\xi\right)\right]+\xi\left(1+\alpha\right)+\left(1-\alpha\right) > 0$ and $u_2 = 1-\alpha\left(1-\xi\right)+\sigma\xi\left(m_0-1\right)+m_0\left[m_0\sigma\left(1-\xi\right)\left(1-\alpha\right)+\alpha\left(1-\xi\right)\left(1+\sigma\right)+1+\xi-3\right]\sigma$.

CASE $\sigma < 0$: Since $\partial \dot{c}/\partial k$ is negative at the steady state, $|A| < 0$ and we have a saddle path stability.

CASE $\sigma > 0$: We need to consider two cases:

A1) If $\sigma m_0 < 1$ then $|A| < 0$ and all the steady states are saddle path stable. Mobility costs should not be too high when workers are getting substitutes.

A2) If $\sigma m_0 > 1$ when the steady state level of labor mobility is $x_{SS} < \left(1-\sigma\right)/S\left(\sigma m_0-1\right)$, then it is saddle path stable. Otherwise, we have to check the trace, which is positive if $u_2 > 0$. Hence, if[2]

$$\sigma < \frac{\xi+2m_0+\left[1-\alpha\left(1-\xi\right)\right]}{\xi+2m_0+m_0\left[1-\alpha\left(1-\xi\right)\right]},$$

the $tr(A) > 0$ and the system is unstable. We assume hereinafter that this condition holds.

Next, we show whether the steady states we found are stable.

a) If $q^{\frac{1}{\sigma}}S^{\frac{1}{\sigma}} > (\delta+\rho)^{\frac{1}{\alpha}}S^2/(1-\alpha)^{\frac{1}{\alpha}}N$, we had two possible cases. First, when there is only one steady state then we are in the tangency case where $x_{SS} = \hat{x}$. Then, $|A| = 0$. Second, when there are two steady states we have $\hat{x} > \left(1-\sigma\right)/S\left(\sigma m_0-1\right)$, what implies that $x_{SS_1} < \hat{x}$ and $x_{SS_2} > \hat{x}$. Therefore, the SS_2 is unstable. We can prove that the SS_1 is also unstable by proving that $x_{SS_1} > \left(1-\sigma\right)/S\left(\sigma m_0-1\right)$. If $x_{SS_1} < \left(1-\sigma\right)/S\left(\sigma m_0-1\right)$ then equation B.4 satisfies that $LHS(\left(1-\sigma\right)/S\left(\sigma m_0-1\right)) > RHS(\left(1-\sigma\right)/S\left(\sigma m_0-1\right))$. This condition implies that

$$\frac{(\delta+\rho)^{\frac{1}{\alpha}}S^2}{(1-\alpha)^{\frac{1}{\alpha}}N} > (qS)^{\frac{1}{\sigma}}\left(\frac{m_0-1}{m_0(1-\sigma)}\right)^{\frac{1}{\sigma}}\frac{1-\sigma}{\sigma(m_0-1)}. \quad \text{(C.1)}$$

[1] For the algebraic computations we use equations 2.8 evaluated at the steady state and 3.4, equation 3.8, and the following relations:

$$k^{(\xi-1)\alpha}\left(1+S\frac{q^{\frac{1}{1-\sigma}}}{m_0^{\frac{\sigma}{1-\sigma}}}k^{\frac{\sigma(1-\xi)}{1-\sigma}}\right)^{\frac{\alpha}{\sigma}} \equiv \left(k^{(\xi-1)\sigma}+qSx^{\sigma}\right)^{\frac{\alpha}{\sigma}},$$

$$\left(1+S\left(\frac{q}{m_0}\right)^{\frac{1}{1-\sigma}}k^{\frac{\sigma(1-\xi)}{1-\sigma}}\right) \equiv 1+Sx.$$

[2] We have applied the fact that $\sigma m_0 > 1$.

On the other hand, since $\hat{x} > (1-\sigma)/S(\sigma m_0 - 1)$ then the LHS of equation B.5 must be larger than the RHS valued at $(1-\sigma)/S(\sigma m_0 - 1)$, which implies that

$$\frac{(\delta+\rho)^{\frac{1}{\alpha}} S^2}{(1-\alpha)^{\frac{1}{\alpha}} N} < (qS)^{\frac{1}{\sigma}} \left(\frac{m_0 - 1}{m_0(1-\sigma)}\right)^{\frac{1-\sigma}{\sigma}} \frac{1}{\sigma m_0}. \tag{C.2}$$

Since $[(m_0-1)/m_0(1-\sigma)]^{(1-\sigma)/\sigma}/\sigma m_0 < [(m_0-1)/m_0(1-\sigma)]^{1/\sigma}(1-\sigma)/\sigma(m_0 - 1)$ always holds because $m_0 > 1$, the two conditions above cannot hold simultaneously. Hence, $x_{SS_1} > (1-\sigma)/S(\sigma m_0 - 1)$.

b) If $q^{\frac{1}{\sigma}} S^{\frac{1}{\sigma}} = (\delta+\rho)^{\frac{1}{\alpha}} S^2/(1-\alpha)^{\frac{1}{\alpha}} N$, we had one steady state. It is saddle path stable if equation C.1 is satisfied. We know that equation C.2 is satisfied since $\widehat{x} = \infty$. Hence, this equilibrium cannot be saddle path stable. Instead, it is unstable.

c) If $q^{\frac{1}{\sigma}} S^{\frac{1}{\sigma}} < (\delta+\rho)^{\frac{1}{\alpha}} S^2/(1-\alpha)^{\frac{1}{\alpha}} N$, we had one steady state (non-tangency). This steady state is saddle path stable if equation B.4 satisfies that $LHS((1-\sigma)/S(\sigma m_0 - 1)) > RHS((1-\sigma)/S(\sigma m_0 - 1))$. Otherwise, it is unstable.

C.2. C.2. Perfect substitution: $\sigma = 1$

1) When firms hire both types of workers, then by eq. 3.8 we have a steady state where

$$k = \left(\frac{m_0}{q}\right)^{\frac{1}{1-\xi}}.$$

Thus, substituting in the consumption and physical capital growth equations we obtain

$$\dot{c} = \frac{c}{\theta}\left[(1-\alpha)\left(\frac{1}{m_0} + Sx\right)^{\alpha}\left(S + S^2 x\right)^{-\alpha} q^{\alpha} N^{\alpha} - \delta - \rho\right], \tag{C.3}$$

$$0 = \frac{N^{\alpha} q \left(\frac{m_0}{q}\right)^{\frac{1}{1-\xi}}}{S^{\alpha}} \frac{\left[\frac{1}{m_0} + (1-\alpha+\frac{\alpha}{m_0})Sx\right]}{(1+Sx)^{\alpha}\left(\frac{q}{m_0} + qSx\right)^{1-\alpha}} - \frac{N}{S}c - \delta\left(\frac{m_0}{q}\right)^{\frac{1}{1-\xi}}. \tag{C.4}$$

The first equation characterizes the dynamics of the economy, but notice that we have it in terms of x. The second equation is equal to zero because k is constant. Equation C.4 gives us the relationship between c and x. To analyze the stability of the steady state, we need to check the sign of

$$\frac{d\dot{c}}{dc} = \frac{\partial \dot{c}}{\partial c} + \frac{\partial \dot{c}}{\partial x}\frac{dx}{dc}.$$

The derivatives evaluated at the steady state are $\partial\dot{c}/\partial c = 0$ and

$$\frac{\partial \dot{c}}{\partial x} = \frac{\alpha(1-\alpha)cq^{\alpha}N^{\alpha}}{\theta S^{\alpha}}\left(\frac{\left(\frac{1}{m_0} + Sx\right)}{(1+Sx)}\right)^{\alpha-1}\left[\frac{S(m_0-1)}{m_0(1+Sx)^2}\right] > 0.$$

Therefore, the sign of $d\dot{c}/dc$ depends on the sign of dx/dc. Using the implicit function theorem we have

$$\frac{\partial x}{\partial c} = -\frac{\frac{\partial (C.4)}{\partial c}}{\frac{\partial (C.4)}{\partial x}},$$

where

$$\begin{aligned}
\frac{\partial (C.4)}{\partial c} &= -\frac{N}{S} < 0, \\
\frac{\partial (C.4)}{\partial x} &= N^{\alpha} q \left(\frac{m_0}{q}\right)^{\frac{1}{1-\xi}} \frac{(1-\alpha+\frac{\alpha}{m_0})S\left(\frac{q}{m_0}+qSx\right)}{S^{\alpha}(1+Sx)^{\alpha}\left(\frac{q}{m_0}+qSx\right)^{2-\alpha}} - \\
&\quad \frac{N^{\alpha} q \left(\frac{m_0}{q}\right)^{\frac{1}{1-\xi}} \left[\alpha(1+Sx)^{-1} S\left(\frac{q}{m_0}+qSx\right)+(1-\alpha)qS\right]}{S^{\alpha}(1+Sx)^{\alpha}\left(\frac{q}{m_0}+qSx\right)^{2-\alpha}\left[\frac{1}{m_0}+(1-\alpha+\frac{\alpha}{m_0})Sx\right]^{-1}}.
\end{aligned}$$

Hence, the sign of this derivative is given by the sign of $\partial(C.4)/\partial x$. In the steady state, from equations 2.8, 3.4 and 3.7 we know that

$$\left(\frac{q}{m_0}+qSx\right) = \left(\frac{\delta+\rho}{1-\alpha}\right)^{\frac{1}{\alpha}} \frac{S}{N}(1+Sx). \tag{C.5}$$

After substituting and rearranging we obtain

$$\begin{aligned}
\frac{\partial (C.4)}{\partial x} &= \frac{N^{\alpha} q(1-\alpha)\left[\left(\frac{\delta+\rho}{1-\alpha}\right)^{\frac{1}{\alpha}}\frac{S}{N}-\frac{q}{m_0}\right]}{\left(\frac{q}{m_0}\right)^{\frac{1}{1-\xi}} S^{\alpha-1}(1+Sx)^{\alpha}\left(\frac{q}{m_0}+qSx\right)^{2-\alpha}} + \\
&\quad \frac{N^{\alpha} q(1-\alpha+\frac{\alpha}{m_0})Sx(1-\alpha)\left[\left(\frac{\delta+\rho}{1-\alpha}\right)^{\frac{1}{\alpha}}\frac{S}{N}-q\right]}{\left(\frac{q}{m_0}\right)^{\frac{1}{1-\xi}} S^{\alpha-1}(1+Sx)^{\alpha}\left(\frac{q}{m_0}+qSx\right)^{2-\alpha}}.
\end{aligned}$$

The derivative is positive as long as the following condition holds:

$$\left(\frac{\delta+\rho}{1-\alpha}\right)^{\frac{1}{\alpha}} \frac{S}{N} > \frac{q}{m_0}.$$

Using equation C.5, this condition can be rewritten as

$$\frac{\left(\frac{q}{m_0}+qSx\right)}{(1+Sx)} > \frac{q}{m_0},$$

which is always true because of $m_0 > 1$. Thus, the steady state is unstable.

2) In the steady state with no labor mobility the dynamic system becomes

$$\begin{aligned}
\dot{c} &= \frac{c}{\theta}\left[(1-\alpha)\left(\frac{N}{S}\right)^{\alpha} k^{(\xi-1)\alpha} - \delta - \rho\right], \\
\dot{k} &= \left(\frac{N}{S}\right)^{\alpha} k^{(\xi-1)\alpha+1} - \frac{N}{S}c - \delta k.
\end{aligned}$$

Since the derivatives evaluated at the steady state are $\partial\dot{c}/\partial c = 0$, $\partial\dot{k}/\partial c = -N/S$ and

$$\frac{\partial\dot{c}}{\partial k} = -\frac{c}{\theta}(1-\alpha)(1-\xi)\,\alpha\left(\frac{N}{S}\right)^{\alpha} k^{(\xi-1)\alpha-1} < 0,$$

we have saddle path stability.

3) To check the stability of the BGP with full labor mobility note that equations 2.8, 3.4 and 3.7 become

$$(1-\alpha)(qS)^{\frac{\alpha}{\sigma}}\left(\frac{N}{S^2}\right)^{\alpha} = r+\delta = \theta\frac{\dot{c}}{c}+\rho+\delta,$$

so that we have an AK model without transition.